Cambridge Assessment Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2019

Examiner's Report

Hints and Solutions

Mark Scheme

Updated January 2020

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Examiner's Report

Introduction

There was a significant rise in the total entry this year with an increase of nearly 8.5% on 2018. One question was attempted by over 90%, two others were very popular, and three further questions were attempted by 60% or more. No question was generally avoided and even the least popular attracted more than 10% of the candidates. 88% restricted themselves to attempting no more than 7 questions, and only a handful, but not the very best, scored strongly attempting more than 7 questions.

Question 1

This was the third most popular question attracting 83% of candidates, though it was only the 6th most successful with an average mark just under 8/20, and no candidate achieved full marks on it. Small mistakes when starting the question often resulted in very different differential equations and hence solutions which lost lots of marks. Some candidates simply did not know how to solve a differential equation and they appeared to waste a lot of time. Drawing the spiral in part (i), lots of candidates guessed that the extrema occurred on the axes losing substantial marks. Candidates should have spent a few minutes more sketching their graphs as these sketches were the main feature of the question and a lot of candidates forgot to mark salient points. Attempting the final sketch, only a handful considered the behaviour for large t.

Question 2

The most popular question, it was also the most successful with an average score of over 11/20 and many fully correct solutions. Most candidates that noticed that the equation for $f(x+y)$ implied $f(x) = 0\ \forall x$, or $f(0) = 1$ correctly eliminated the former, but quite a few did not realise that it was a possibility to consider. Nearly every candidate successfully found

$f'(x) = f(x)\lim_{h\to 0}\left(\frac{f(h)-1}{h}\right)$ and most also proceeded correctly from there to find the required differential equation. Finding f(x) was generally successful although some did not check the boundary conditions. In part (ii), there were fewer issues demonstrating that $g(0) = 0$ than there had been with $f(0) = 1$ in part (i). The simplification in order to find the limit to obtain $g'(x)$ was usually successful. Solution of that differential equation was often well done, either using partial fractions or as a hyperbolic function, although some mistakenly identified the solution as a tan function.

Question 3

This was the 6th most popular question, but the least successful of those six, indeed one of the four least successful in the whole paper scoring just under a quarter of the marks. Barely a handful of attempts scored full marks. Candidates frequently overlooked the last result in (i), or merely wrote **A=I** , presumably assuming that was hardly worth any marks. Those who got to part (iii) typically did well on it, even if they had done poorly on parts (i) and (ii). Candidates often omitted or struggled to deal with cases. Common pitfalls were unnecessary division by zero without considering if the denominator were zero, thinking that the question stated that all invariant points lie on a line leads to claims that **A** is non-linear or doesn't exist, as mentioned failure to justify that **A=I**, and use of det(**A-I**)=0 with no justification.

Question 4

Two thirds of candidates attempted this scoring, on average about half marks. The first part was often well answered by those that used the Vieta equations, though a common error was to divide by a potential zero and therefore omit one of the solutions. Those that substituted the three roots into the polynomial equation encountered equations that were more difficult to solve, and the method yielded additional solutions which were often not rejected, so generally those taking this approach did much less well. Part (ii) had many good answers from squaring the sum of roots equation, though a common error was made with inaccurate summation notation. Many argued the deduction correctly but likewise many others assumed the a s were non-zero without a reason. Part (iii) was found more difficult. Those candidates that did not realise the significance of the previous deduction were rarely successful and others gave the correct answer, but with no explanation. Those making the inductive argument to factorise powers of x rarely justified that the remaining polynomial was likewise reflexive.

Question 5

A little more popular than question 1, attempts were only a little less successful than those for question 4. The effects and consequences of the typographical error in the substitution for part (ii) are dealt with in https://www.admissionstesting.org/Images/552575-step-2019-paper-3-question-5-protocol-summary.pdf

The majority of candidates successfully drew the graph for (i), but common errors were failures to consider asymptotic behaviour and label appropriately. The vast majority used the incorrectly suggested substitution and proceeded as far as possible on the first result of (ii) using it; a few realised at this point that there was an error and then obtained the correct result. Candidates who moved on to the evaluation, scored strongly using the quoted result, even if they had not obtained it, and appreciated that the limits were needing to be changed. Most attempting the second evaluation of (ii) obtained full marks on that part as they successfully demonstrated that it was the same answer as the previous evaluation. Few attempted part (iii), and most that were successful made substitutions not involving using the previous part of the question.

Question 6

Half the candidates tried this question, but it was one of the four least successfully attempted. The majority successfully demonstrated that the locus of P was a circle with the correct centre and radius, but few made further significant progress. They usually substituted for z in terms of w but then failed to rearrange into the necessary form. If they did achieve this, then they were able to score most of the marks up until the very last part which required careful justification to earn full marks.

Question 7

Attempted by two thirds, the mean score was only about one third marks. Part (i) was not very well answered with many appearing to guess one or both of the solutions without managing to factorise, or equivalent. By contrast, part (ii) (a) was generally well-answered. Most candidates saw how to do this correctly, though a few tried to treat it as a polynomial in y and take that discriminant which got them nowhere. In part (b), most candidates did not realise what was expected of them so, for example, many wrote $x \to \infty, y \to \infty$ or found the x and y intercepts. Almost all candidates saw what was needed for part (c), however, many made small algebraic errors or didn't check all the cases; the most common error was failure to eliminate the origin. The sketch was generally badly answered, often owing to errors in previous parts or failure to use correctly the information already obtained. Candidates did appreciate what was needed for part (iii) and did this well.

Question 8

The least popular of the Pure questions it was attempted by about a quarter of the candidates. It was generally found quite challenging with many attempts receiving little or no credit and so it was one of the four least successfully attempted questions. Some struggled to handle the vectors in part (i) with some attempting to divide by vectors or confusing cross and dot products, or normals and tangents, though these sorts of errors were rare. Much more common were errors arising from incorrect signs in the projections onto the base vectors **i**, **j**, and **k** or failure to recognise whether the angle being calculated was θ or $\pi - \theta$. In general, most candidates who took the time to establish a clear vector space set-up did rather well, not just in (i) but in (ii) as well. Most who attempted the first result of (ii) did so successfully. The most challenging part of the question was found to be obtaining the expression for $\cos^2 \varphi$ which required several small insights relating to trigonometric identities and a fair amount of calculation. A pleasing proportion attempting the calculation did so successfully but again a sound coordinate based set-up rendered it manageable. A number of candidates failed to justify properly the, often elementary, steps for the final part losing credit by so failing to do. Overall the question was largely algebraic rather than geometric, but the best solutions used the interplay between these two aspects to great effect.

Question 9

Although this was the least frequently attempted question, it was tried by just over 10% scoring on average marginally under one third marks. The position and velocity in (i) were successful for most, and those considering momentum (or centre of mass) found the next result easy, although those considering forces struggled. Most struggled with (ii), for whilst considering energy, typically they forgot the kinetic energy of the hemisphere. Part (iii) evoked a number of approaches, which were usually unsuccessful, and most could not make use of the suggested method. Even candidates who were unsuccessful with the rest of the question were able to obtain the cubic equation, though hardly any could justify the final inequality.

Question 10

Comfortably the most popular applied question on the paper with two fifths of candidates trying it, it was also one of the most successfully attempted on the whole paper with an average score just shy of half marks. A significant number of candidates struggled to set up the problem correctly, but those that did generally obtained the first result of (i). Then common mistakes were using Newton's Law of Impact in this part and failure to express w in terms of the specified variables. In part (ii), most candidates correctly applied Newton's Law of Impact, but depending on their expression for w in (i) had varying levels of success obtaining the required expression. A lot of candidates did manage to gain most marks in the final part even if they had struggled earlier. Alongside trigonometric and algebraic mistakes, a common mistake was failing to express $\tan\theta$ in terms of $\tan\alpha$, as against other trigonometric ratios, and e. The commonest approach for the final result was to use differentiation, although a few candidates successfully used the AM-GM inequality. However, fewer than a handful of candidates justified the value being a global as against local maximum.

Question 11

The least successful question with a mean score of just under one quarter marks, it was attempted by a fifth of the candidates. Many candidates assumed for (i) that the number of customers taking sand would follow a Poisson distribution without giving a proof and some thought that checking the mean equalled the variance was sufficient. For (ii), some assumed $E[f(x)] = f(E[X])$ which may have been with an eye to the 'show that' so e.g. '$Y \sim Po(\lambda p)$, $e^{-k\lambda p} = E[e^{-kY}]$'. For (iii), a minority used the law of total probability and were generally successful. A popular approach was to spot that $P(assistant\ gets\ sand) = E[proportion\ of\ sand\ they\ take]$, however few were able to express this as a correct probabilistic statement. In particular, some treated the amount of sand taken/remaining as a deterministic constant equal to its mean. Many candidates struggled to differentiate $ke^{-k\lambda p}$ with respect to k, and few gave a valid justification why the stationary point was a maximum.

Question 12

A quarter of the candidates attempted this scoring on average about half marks, making it the second most successfully attempted question. Candidates using the approaches suggested by the question tended to make good progress. Many produced correct solutions using various combinatorial arguments, some of which were easier to generalise than others and not all were well-explained. In part (ii), a few candidates used arguments not permitted by the wording of the question. In part (iii), some obtained the first result by the nice method $P(A_1 \subseteq A_2) = P(A_1 \cap A_2' = \emptyset)$ which is of course the first result of (ii) but this was not easy to generalise. Several incorrectly assumed $P(A_1 \subseteq A_2 \subseteq A_3) = P(A_1 \subseteq A_2)\, P(A_2 \subseteq A_3)$ "by independence".

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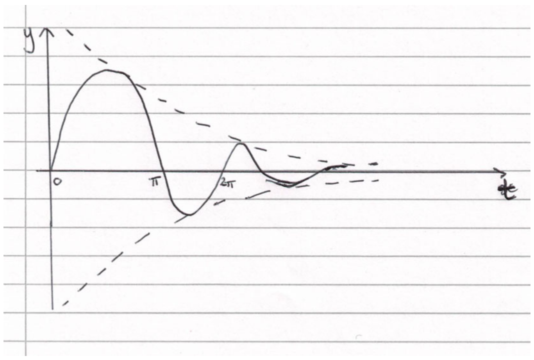
2019

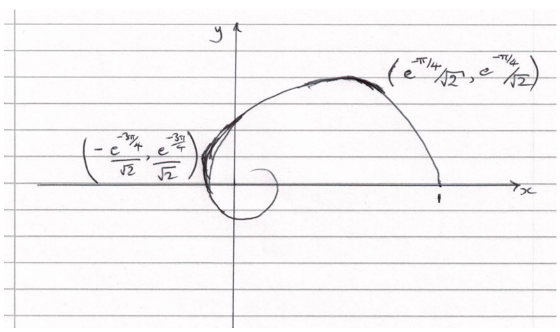
Hints and Solutions

Question 1

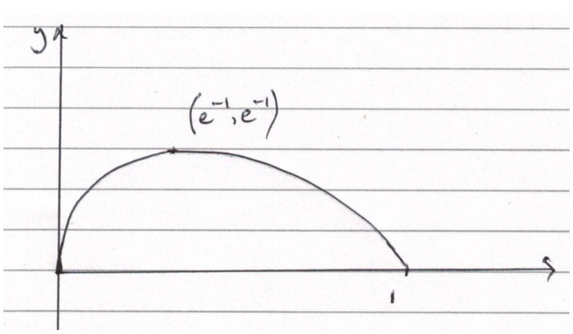
(i) Making y the subject of the first equation, it can be substituted in the second to give a second order differential equation solely in x. Using an auxiliary quadratic equation, this can be solved for x, and then y can be found from the first equation in terms of the same arbitrary constants, which can be evaluated using the initial conditions to give $x = e^{-t}\cos t$ and $y = e^{-t}\sin t$. The sketch of y as a function of t is



Differentiating y and then x with respect to t and in each case equating to zero gives the values of t for which y is greatest and for which x is least, giving the points $\left(\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$ and $\left(-\frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}\right)$ so the sketch is



(ii) In this case, the first equation becomes a separable first order differential equation for x and substituting its solution into the second equation yields a first order differential equation for y which can be solved using an integrating factor. The arbitrary constants can be evaluated using the initial conditions to give $x = e^{-t}$ and $y = te^{-t}$. In order to sketch the path, x and y should both be differentiated with respect to t which indicate that the gradient at $(1,0)$ is -1, that there is a maximum for y at (e^{-1}, e^{-1}) and that as the curve approaches the origin, it approaches the y axis tangentially, so the sketch is



Question 2

(i) Substituting $y = 0$ (and optionally $x = 0$) in the expression for $f(x + y)$ yields two possibilities, the required $f(0) = 1$ or a second which can be dismissed because $f'(0) \neq 0$. Using (*) and the expression for $f(x + y)$ applied for $y = h$ yields a factor $f(x)$ and the limiting expression is as required. Separating variables and applying the initial condition already found yields $f(x) = e^{kx}$. Part (ii) proceeds similarly finding $g(0) = 0$, with a second option dismissed this time owing to the inequality for the modulus of g. The limiting follows that of (i) with a factor this time of $\left(1 - \left(g(x)\right)^2\right)$ giving $g'(x) = k\left(1 - \left(g(x)\right)^2\right)$. Separating variables and then either using partial fractions or using the standard hyperbolic result yields the solution in one of the forms

$g(x) = \frac{e^{2kx}-1}{e^{2kx}+1} = \tanh(kx)$ once the initial condition has been imposed.

Question 3

Writing $A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ and expanding as two linear equations, in each of which the terms in x are on one side of the equation and y on the other, they can then be multiplied together and then rearranged to give the first desired result in (i). The further result requires consideration of the cases $x=0$ and $y=0$ which lead to values for either a and c, or b and d, which both similarly imply the required result. An alternative approach to this is to rewrite the original matrix equation in the form $B\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ and as this does not have a unique solution being true for a line of invariant points, the determinant of B must be zero yielding both of the first two results. The final result of (i) can be obtained by considering $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\mx+k\end{pmatrix}=\begin{pmatrix}x\\mx+k\end{pmatrix}$ with $k\neq 0$ $\forall x$ and $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}k\\y\end{pmatrix}=\begin{pmatrix}k\\y\end{pmatrix}$ with $k\neq 0$ $\forall y$, to evaluate a, b, c and d, and hence conclude that $A=I$. In (ii), a similar approach to that used at the start of the question as the condition if and only if a point is invariant leads to the line $(a-1)x+by=0$ being the same as $cx+(d-1)y=0$ if $(a-1)(d-1)=bc$ and $b\neq 0$. On the other hand, if $(a-1)(d-1)=bc$ and $b=0$ then the cases of $a=1$ or $\neq 1$ can be followed through to give lines $cx+(d-1)y=0$ or the y axis as invariant. Part (iii) can be approached from considering $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\mx+k\end{pmatrix}=\begin{pmatrix}x'\\mx'+k\end{pmatrix}$ with $k\neq 0$. As this is true for all x, considering the case $x=0$ yields an equation in b, d and m, and similarly $x=1$ (of course using say x'' instead of x') yields one in a, c and m. Combining these yields $m(a-1)(d-1)=mbc$ and so the case $m=0$ has to be considered, which returning to the initial line of working for this part and considering for two values of x gives $d=1$ and $c=0$ and hence the desired result.

Question 4

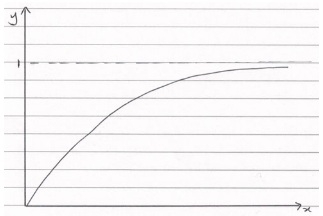
(i) For $n = 1$, it is easy to show that $x - a_1$ is reflexive. For $= 2$, Vieta's equations yield that $a_2 = 0$ and a_1 can take any value giving $x^2 - a_1 x$. For $= 3$, Vieta's equations yield $a_2 = -a_3$ from the sum of roots, and as a consequence that $a_2 = 0$ or $a_3 = 1$ from the sum of products of pairs of roots, leading, after using the third equation to the possibilities $x^3 - a_1 x^2$ and $x^3 + x^2 - x - 1$.

(ii) From the sum of roots Vieta relation a_1 can be eliminated, and the result can then be squared which with a little careful manipulation of the sum of product of roots two at a time Vieta relation yields the desired result. For $n > 3$ completion of the square yields a square on LHS and 1 minus a sum of squares on RHS. That the coefficients are integers can only yield ${a_n}^2 = 1$ for $a_n \neq 0$ and the other coefficients for $r = 3, \dots, n - 1$ are zero, which establishes a contradiction when considering the product of all roots.

(iii) As it has been deduced in (ii) that for any reflexive polynomial of degree greater than 3 that $a_n = 0$, such a polynomial can be factorised by x and then it can be carefully argued that the resulting polynomial is reflexive and so there is an inductive argument. So essentially the solutions are those found in (i) along with those solutions multiplied by arbitrary positive integer powers of x, that is $(x - a_1)x^r$ or $(x + 1)^2(x - 1)x^r$ with $r = 0, 1, 2, \dots$

Question 5

Differentiating f(x) and considering behaviour at the origin and for large x gives the following graph.



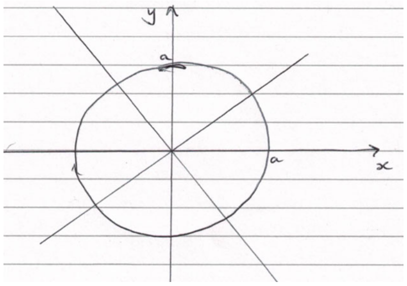
Using the substitution $y = \frac{cx}{\sqrt{x^2+p}}$ in the integral I, it can be seen to simplify to the given result if we choose $= 1$. The first evaluation uses that result with $c = \sqrt{2}$, and $= \sqrt{3}$, giving a result having used the noted standard integral of $\frac{\pi}{3\sqrt{3}}$. Making the substitution, $= \frac{1}{x}$, the second evaluation can be shown to be the same as the first. Returning to making the same general substitution (i.e. $y = \frac{bx}{\sqrt{x^2+p}}$ say) for part (iii),the resulting integral can be seen to be simplified by choice of b and p ($2b^2 = 1$ and $p = -1$) to simplify to the standard integral noted in the question and hence a result of $\frac{\pi}{4}$.

Question 6

The stem is achieved by using the direction given in the question, expansion and rearrangement to obtain the desired result, with a radius r and centre a. To obtain the locus of Q, substituting for z in terms of w, then multiplying through by ww^* and dividing by $aa^* - r^2$ followed by some simplification yields $\left|w - \frac{a^*}{aa^*-r^2}\right|^2 = \frac{r^2}{(aa^*-r^2)^2}$ which represents a circle centre $\frac{a^*}{aa^*-r^2}$ and radius $\left|\frac{r}{aa^*-r^2}\right|$. The second result of (i) is obtained by equating the radii and simplifying. Equating the centres and substituting for the denominator as each of the values derived from the result just found gives $a = \pm a^*$ and so the two possibilities for a can be concluded. In the case that is real, working back to the value that produced it, the radius is smaller than $|a|$ so the circle is centred on the real axis and does not contain the origin, whereas in the other case, the opposite is true so the origin is contained. For part (ii), the given result of part (i) still applies by a repetition of the previous working but with the new substitution, but equating the radii now gives that any a is possible with $|a| = \sqrt{r^2 + 1}$ (or a is zero) so the previous statement regarding a does not apply.

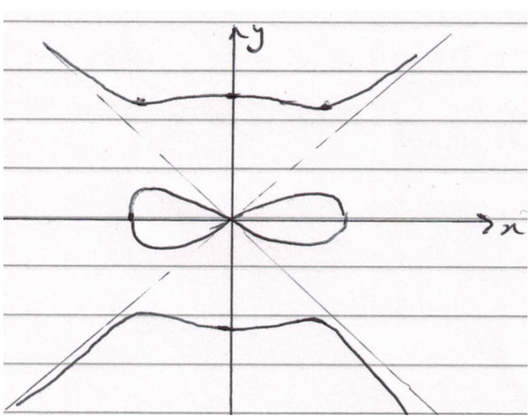
Question 7

(i) Treating both sides of the given equation as biquadratics and completing the squares, or subtracting and factorising the differences of squares, yields the line pair $y = \pm x$ and the circle $x^2 + y^2 = a^2$.



Following the instructions in the question for part (ii) (a), the positive discriminant condition for the biquadratic in x yields a biquadratic inequality for y which readily factorises to give the required result, bearing in mind that y positive ensures that two of the four linear factors are thus positive regardless. In part (b), 'close to the origin' indicates that only the terms of lowest degree need consideration, giving $y \approx \pm\frac{2x}{\sqrt{5}}$ (though the negative is discounted), and 'very far from the origin', the highest degree terms and $y \approx \pm x$ respectively. For (c) differentiating and setting $\frac{dy}{dx}$ equal to zero gives a simple cubic equation in x, which gives $\left(0, \sqrt{5}\right)$, $\left(\sqrt{2}\,,1\right)$, $\left(\sqrt{2}\,,2\right)$ but from (b) not $(0, 0)$ as points where the tangents are parallel to the x axis. Likewise for 'parallel to the y axis' with $\frac{dx}{dy}$, but the working of (a) and (b) restricts the possibilities to just the single one $(2\,, 0)$.

Using all the information gleaned, the sketch for (iii) is merely that for (ii)(c) employing symmetry in both axes, though, in both cases, care should be taken to ensure that the relative gradients near and from the origin reflect the results of (ii) (b).



Question 8

Using the convention of labelling ABCD anticlockwise and labelling the midpoint of AB as M, it is straightforward to find the vector MV as a scalar multiple of a unit vector and from it to deduce the required vector perpendicular to AVB as $\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix}$. Doing the same for BVC yields $\begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix}$ and the scalar product of these generates the result for (i). Labelling the centre of the base ACD as W, simple trigonometry enables lengths MW, BM and BW to be expressed in terms of VW, , β and θ, then the first result of (ii) is using Pythagoras. Using this result and standard trigonometric identities, it is possible to proceed to $\tan^2\varphi$, to $\sec^2\varphi$, and so to $\cos^2\varphi$; doing the comparable steps with α and β in the expression obtained and using the result of (i) where the product appears yields the required expression and considering $(\cos\alpha - \cos\beta)^2 \geq 0$ leads to the inequality. For the deduction, as θ is acute, the factor $1 - \cos\theta$ is positive and so can be cancelled, and also both $\frac{2}{(1+\cos\theta)} > 1$ and $\cos\theta > \cos\theta\cos\theta$. That $\cos^2\varphi \geq \cos^2\theta$, given the positivity of both cosines means the same applies to the unsquared quantities and hence the final result.

Question 9

The expression for the position of the particle is $= (a\sin\theta - s)\boldsymbol{i} + a\cos\theta\,\boldsymbol{j}$, obtained by adding the displacement of the particle relative to the hemisphere to that of the hemisphere, and differentiating this with respect to time gives the first required result in (i). Conserving the horizontal linear momentum of the system (i.e. the particle and the hemisphere together) and rearranging for $\dot{s}$ eliminating the masses in favour of the given variable k obtains the second result. The deduction is achieved by substituting the second result in the first. Part (ii) is obtained by conserving energy for the system, using the results for the speed of the hemisphere, the speed of the particle derived from its velocity obtained in (i), the gravitational potential energy of the particle and eliminating the masses using k. Part (iii) commences by finding $\ddot{\boldsymbol{r}}$ by differentiating $\dot{\boldsymbol{r}}$ from (i), then appreciating that the particle loses contact when $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$, equating the two expressions and taking the scalar product of the equation with the suggested vector. Substituting the required result in the result of (ii) leads to the equation $(k-1)\cos^3\alpha + 3\cos\alpha - 2 = 0$, and the deduction can be made by taking the term of degree 3 to the other side of the equation, and appreciating that in doing so it can be seen to be positive.

Question 10

A good diagram showing Q moving off along the line of centres after the collision and conserving linear momentum perpendicular to that line obtains the first result of (i). The expression for w can be found by conserving momentum perpendicular to the original direction of motion of P, or alternatively in the direction of the line of centres and then substituting for u using the first result; $w = v\frac{\sin\theta}{\sin\alpha}$. The first result of (ii) can be found by applying Newton's Law of Impact and then substituting for w and u using the results of (i). Using this result, expanding using compound angle formulae, and dividing by $\cos\theta\cos^2\alpha$ leads to $\tan\theta = \frac{(1+e)\tan\alpha}{1+2\tan^2\alpha - e}$ having applied trigonometric identities. To obtain the maximum of $\tan\theta$, it is worth simplifying the working by letting $t = \tan\alpha$ and differentiating with respect to t. The maximum value is $\frac{\sqrt{2}(1+e)}{4\sqrt{1-e}}$ (which occurs when $= \sqrt{\frac{1-e}{2}}$) which can be justified as it is the only stationary value, consideration of when $t = 0$, $t \to \infty$ and that $\tan\theta > 0$ for all t.

Question 11

To satisfy (i), it is necessary to find the probability that a number, say r, take sand which can be calculated by summing the product of the Poisson probability that a number of customers arrive and the binomial probability that r of that number take the offer; factorising out all the terms not involved in the summation index leaves a sum that can be recognised as an exponential function, and the result then follows. The mass taken by n customers can be expressed as a GP with first term kS and common ratio $(1-k)$ so the case $k=0$ needs considering separately, and then the expectation can be found by summing the product of these masses and the probabilities using the result of (i); each term naturally splits into two parts giving two exponential sums whose difference leads to the given result. Using the working from (i) and (ii), if r customers take the sand, the amount the assistant takes is $S(1-k)^r$ and so the probability the assistant takes the golden grain in that case can be shown to be $k(1-k)^r$. Summing over r the product of Poisson probabilities and that just found gives the probability the assistant takes the golden grain as $e^{-k\lambda p}$. In the case $k=0$, no sand is taken by anyone at all, so the answer is zero as per the formula, and as $\rightarrow 1$, the only way the assistant can get the grain is if no customer takes the sand so the probability approaches $e^{-\lambda p}$. To maximise, differentiation with respect to k yields a stationary value when $k=\frac{1}{\lambda p}$ which the given condition ensures is less than 1, and justification that this is a maximum can be shown by considering the gradient either side of this value or by using the second derivative.

Question 12

The result required in the stem can be derived by considering that for each subset, each integer can be an element of it or not leading to the number of possibilities stated. An alternative is to consider the sum of the number of subsets there are with r elements written as binomial coefficients and appreciating that this sum is $(1+1)^n$. $P(1 \in A_1) = \frac{1}{2}$ for (i) as 1 is equally likely to be or not in the set A_1. A similar approach for part (ii) yields $P(t \in A_1 \cap A_2) = \frac{1}{4}$ for any particular integer t and from that, the complement extended to all n integers gives the required result. The other two solutions are similarly $P(\, A_1 \cap A_2 \cap A_3 = \emptyset\,) = \left(\frac{7}{8}\right)^n$ and $P(\, A_1 \cap A_2 \cap \ldots \cap A_m = \emptyset\,) = \left(1 - \frac{1}{2^m}\right)^n$.

For (iii), considering that for any particular integer t, if $A_1 \subseteq A_2$ then $t \in A_1 \cap A_2$, $t \in A_1' \cap A_2$ or $t \in A_1' \cap A_2'$, gives $P(A_1 \subseteq A_2) = \left(\frac{3}{4}\right)^n$. The same approach yields the other two results as $P(A_1 \subseteq A_2 \subseteq A_3) = \left(\frac{4}{8}\right)^n = \left(\frac{1}{2}\right)^n$ and $P(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m) = \left(\frac{m+1}{2^m}\right)^n$.

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Mark Scheme

1. (i) $\dot{x} = -x - y$

So $y = -\dot{x} - x$

As $\dot{y} = x - y$, $-\ddot{x} - \dot{x} = x + \dot{x} + x$

$$\ddot{x} + 2\dot{x} + 2x = 0$$

M1

AQE $\lambda^2 + 2\lambda + 2 = 0$, $\lambda = -1 \pm i$ so $x = e^{-t}(A\cos t + B\sin t)$ **M1 A1 cao**

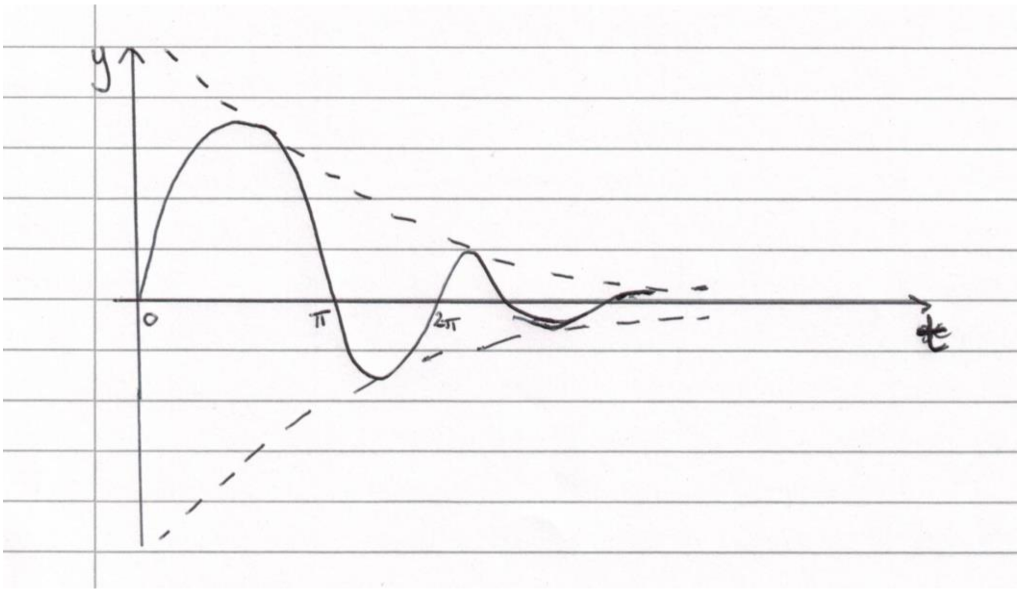
[Alternatively, $x = \dot{y} + y$ leading to $\ddot{y} + 2\dot{y} + 2y = 0$ and $y = e^{-t}(C\cos t + D\sin t)$ etc]

So $y = e^{-t}(A\cos t + B\sin t) - e^{-t}(-A\sin t + B\cos t) - e^{-t}(A\cos t + B\sin t)$

$$= e^{-t}(A\sin t - B\cos t)$$

$t = 0,\ x = 1 \Rightarrow A = 1$ and $t = 0,\ y = 0 \Rightarrow B = 0$

So $x = e^{-t}\cos t$ and $y = e^{-t}\sin t$ **M1 A1 cao**



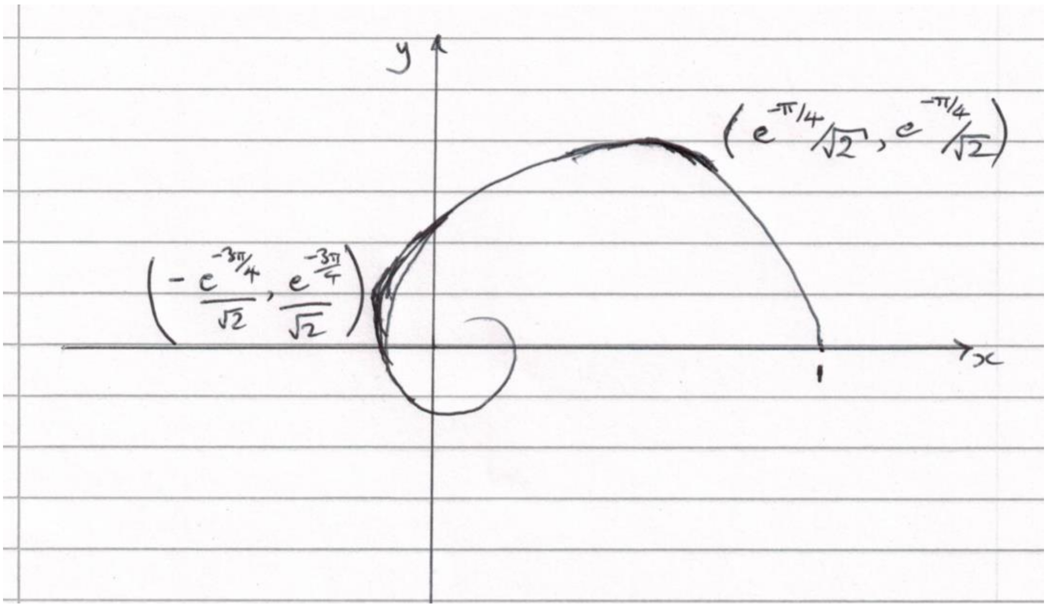
G1 G1 **[7]**

y is greatest when $\dot{y} = 0$ $\Rightarrow x = y$ $\Rightarrow$ $\tan t = 1$ $\Rightarrow t = \frac{\pi}{4} + n\pi$

Hence $t = \frac{\pi}{4}$ thus $\left(\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$ **M1**

x is least when $\dot{x} = 0$ $\Rightarrow x = -y$ $\Rightarrow \tan t = -1$ $\Rightarrow t = \frac{3\pi}{4} + n\pi$

Hence $t = \frac{3\pi}{4}$ thus $\left(-\frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}\right)$ **M1**



G1 G1 G1 **[5]**

(ii) $\dot{x} = -x \quad \Rightarrow x = Ae^{-t}$

$t = 0,\ x = 1\ \Rightarrow A = 1\ \Rightarrow x = e^{-t}$ **M1**

So $\dot{y} + y = e^{-t}$

The integrating factor is e^t . Thus $e^t y = \int 1dt = t + c$

$y = (t + c)e^{-t}$ **M1**

$t = 0,\ y = 0\ \Rightarrow c = 0$ so $y = te^{-t}$ **A1 cao**

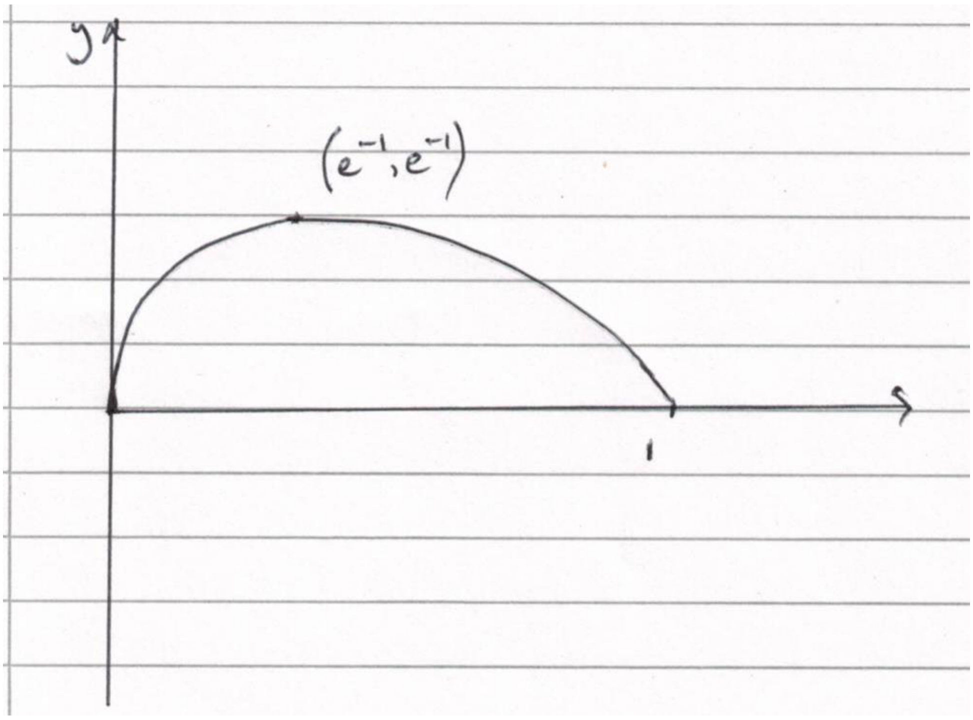
$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{(1-t)e^{-t}}{-e^{-t}} = t - 1$ so there is a maximum at (e^{-1}, e^{-1}) **M1**

As $t \to \infty$, $\frac{dy}{dx} \to \infty$ **M1**

Alternatively,

As $\dot{y} = x - y$, $\ddot{y} = \dot{x} - \dot{y} = -x - \dot{y}$

So $\ddot{y} + 2\dot{y} + y = 0$ yielding $y = (At + B)e^{-t}$ **M1**



G1 G1 G1 **[8]**

2. (i) Let $y = 0$, $f(x+0) = f(x)f(0)\ \forall x$, so $f(x) = f(x)f(0)\ \forall x$ **M1**

Thus $f(x)\big(1 - f(0)\big) = 0\ \forall x$. Therefore, $f(x) = 0\ \forall x$, or $f(0) = 1$ **M1**

If $f(x) = 0\ \forall x$ then $f'(x) = 0\ \forall x$ but $f'(0) = k \neq 0$ so $f(0) = 1$ **A1* cso**

$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{f(x)f(h)-f(x)}{h}\right) = f(x)\lim_{h\to 0}\left(\frac{f(h)-1}{h}\right)$ **M1**

But $\lim_{h\to 0}\left(\frac{f(h)-1}{h}\right) = \lim_{h\to 0}\left(\frac{f(0+h)-f(0)}{h}\right) = f'(0) = k$ so $f'(x) = k\,f(x)$ **M1 A1* cso**

Thus

$$\frac{f'(x)}{f(x)} = k$$

Integrating

$$\ln\big(f(x)\big) = kx + c$$

M1

So $f(x) = e^{kx+c} = Ae^{kx}$. As $f(0) = 1$, $A = 1$, so $f(x) = e^{kx}$ **M1 A1 cao [9]**

(ii) Let $y = 0$,

$$g(x+0) = \frac{g(x)+g(0)}{1+g(x)g(0)}\ \forall x$$

M1

Thus $g(x) + \big(g(x)\big)^2 g(0) = g(x) + g(0)\ \ \forall x$

So $\left(\big(g(x)\big)^2 - 1\right)g(0) = 0\ \ \forall x$ **M1**

As $|g(x)| < 1$, $\big(g(x)\big)^2 - 1 \neq 0$, and thus $g(0) = 0$ **A1**

$$g'(x) = \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{g(x)+g(h)}{1+g(x)g(h)} - g(x)}{h}\right) = \lim_{h\to 0}\left(\frac{g(x)+g(h)-g(x)\big(1+g(x)g(h)\big)}{h\big(1+g(x)g(h)\big)}\right)$$

M1

$$= \lim_{h\to 0}\left(\frac{g(h)\left(1-\big(g(x)\big)^2\right)}{h\big(1+g(x)g(h)\big)}\right) = \left(1-\big(g(x)\big)^2\right)\lim_{h\to 0}\left(\frac{g(h)}{h\big(1+g(x)g(h)\big)}\right)$$

M1

$$\lim_{h\to 0}\left(\frac{g(h)}{h\big(1+g(x)g(h)\big)}\right) = \lim_{h\to 0}\left(\frac{g(h)/h}{\big(1+g(x)g(h)\big)}\right) = \lim_{h\to 0}\left(\frac{g(h)}{h}\right) = g'(0) = k$$

M1 A1

Thus $g'(x) = k\left(1-\big(g(x)\big)^2\right)$ **A1**

$$\frac{g'(x)}{\left(1-\left(g(x)\right)^2\right)}=k$$

Integrating,

$$\tanh^{-1}\left(g(x)\right)=kx+c$$

M1

$$g(x)=\tanh(kx+c)$$

As $g(0)=0$, $\tanh(c)=0$ and so $c=0$ **M1**

Thus

$$g(x)=\tanh(kx)$$

A1 **[11]**

[Alternatively, integrating having used partial fractions,

$$\frac{1}{2}\ln\left(\frac{1+g(x)}{1-g(x)}\right)=kx+c$$

M1

As $g(0)=0$, $c=0$

$$\frac{1+g(x)}{1-g(x)}=e^{2kx}$$

M1

and so

$$g(x)=\frac{e^{2kx}-1}{e^{2kx}+1}$$

A1 **[3]**

3. (i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \;\Rightarrow\; ax + by = x \quad cx + dy = y$ **M1**

$(a-1)x = -by$

$-cx = (d-1)y$

Thus $(a-1)x(d-1)y = bycx$, that is $(a-1)(d-1)xy - bcxy = 0$

So $\big((a-1)(d-1) - bc\big)xy = 0$ **M1**

Thus $\big((a-1)(d-1) - bc\big) = 0$ or $x = 0$ or $y = 0$ **M1**

If L_1 is $x = 0$, then both $by = 0$ and $dy = y$, $\forall y$

Thus, $b = 0$ and $d = 1$, meaning that $\big((a-1)(d-1) - bc\big) = 0$ **E1**

Similarly, if L_1 is $y = 0$, then both $cx = 0$ and $ax = x$, $\forall x$

Then $c = 0$ and $a = 1$, meaning that $\big((a-1)(d-1) - bc\big) = 0$

In all three cases, $(a-1)(d-1) = bc$ **E1**

Alternatively,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \;\Rightarrow\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ **M1**

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

That is $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **M1**

As this is true for a line of invariant points, it does not have a unique solution and so **E1**

$det\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} = 0$ **M1**

and so $\big((a-1)(d-1) - bc\big) = 0$ which implies both $\big((a-1)(d-1) - bc\big)xy = 0$ and $(a-1)(d-1) = bc$ **E1**

If L_1 does not pass through the origin then either L_1 is a) $y = mx + k$ with $k \neq 0$

or b) $x = k$ with $k \neq 0$ **E1**

For a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} x \\ mx + k \end{pmatrix} \quad \forall x$

Thus $ax + b(mx + k) = x$ and $cx + d(mx + k) = mx + k$

As these apply for all x, and $k \neq 0$, $bk = 0$ which implies $b = 0$ and $a + bm = 1$ and thus $a = 1$

Also $dk = k$ implying $d = 1$ and $c + dm = m$ which thus gives $c = 0$ **M1**

For b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k \\ y \end{pmatrix} = \begin{pmatrix} k \\ y \end{pmatrix}$ $\forall y$

Thus $ak + by = k$ and $ck + dy = y$

So $ak = k$ implying $a = 1$ and $b = 0$ and $ck = 0$ implying $c = 0$ and $d = 1$ **M1**

Thus $\boldsymbol{A} = \boldsymbol{I}$ **A1** **[9]**

(ii) If $(a-1)(d-1) = bc$ and $b \neq 0$

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ is an invariant point iff $ax + by = x$ and $cx + dy = y$ **M1**

That is $(a-1)x + by = 0$, $(a-1)(d-1)x + b(d-1)y = 0$, $bcx + b(d-1)y = 0$ and so $cx + (d-1)y = 0$ as required. **E1**

The line of invariant points is thus $(a-1)x + by = 0$ which is $cx + (d-1)y = 0$ **A1**

If $(a-1)(d-1) = bc$ and $b = 0$ then $a = 1$ or if $a \neq 1$, $d = 1$

$a = 1$, $\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ cx + dy \end{pmatrix}$ so points on $cx + (d-1)y = 0$ are invariant. **B1**

$a \neq 1$, $\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ cx + y \end{pmatrix}$ so points on $x = 0$ are invariant. **B1** **[5]**

(iii) L_2 is an invariant line implies $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} x' \\ mx' + k \end{pmatrix}$ and as L_2 does not pass through the origin $k \neq 0$ **M1**

So $ax + b(mx + k) = x'$ and $cx + d(mx + k) = mx' + k$

As these are true for all x, true for $x = 0$ and thus $bk = x'$ and $dk = mx' + k$ giving

$dk = mbk + k$ and as $k \neq 0$, $d = mb + 1$ **E1**

Similarly, for $x = 1$ and thus $a + b(m + k) = x''$ and $c + d(m + k) = mx'' + k$

So $c + d(m + k) = m\big(a + b(m + k)\big) + k$

$c + dm + dk = ma + (m + k)(d - 1) + k$

$c + dm + dk = ma + dm + dk - m - k + k$

$c = ma - m$ **E1**

So $m(a-1) = c$ and $d - 1 = mb$

Hence, multiplying these $m(a-1)(d-1) = mbc$ **E1**

Thus, if $m \neq 0$, $(a-1)(d-1) = bc$

If $m = 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ k \end{pmatrix} = \begin{pmatrix} x' \\ k \end{pmatrix}$ giving $ax + bk = x'$ and $cx + dk = k$ **E1**

As these must be true for all x, when $x = 0$, $dk = k$ giving $d = 1$ and so for choosing any $x \neq 0$, we find $c = 0$. Thus $(a-1)(d-1) = 0$ and $bc = 0$ giving $(a-1)(d-1) = bc$

E1 **[6]**

4. (i) Degree 1, $x - a_1 = 0$ has root $x = a_1$ and so $x - a_1$ is reflexive. **B1**

Degree 2, $x^2 - a_1 x + a_2 = 0$ has to have roots a_1 and a_2 to be reflexive.

Thus $a_1{}^2 - a_1 a_1 + a_2 = 0$ (giving $a_2 = 0$) and $a_2{}^2 - a_1 a_2 + a_2 = 0$ **M1** which is consistent. Thus, $x^2 - a_1 x \ (+\ 0)$ **B1**

Alternatively

$a_1 = a_1 + a_2$ and $a_2 = a_1 a_2$ giving $a_2 = 0$ and consistent for any a_1 **M1**

Thus, $x^2 - a_1 x \ (+\ 0)$ **B1**

Degree 3, $x^3 - a_1 x^2 + a_2 x - a_3 = 0$

$a_1 = a_1 + a_2 + a_3$

$a_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$

$a_3 = a_1 a_2 a_3$ **M1**

The first equation implies that $a_2 + a_3 = 0$ **M1** or in other words $a_2 = -a_3$

This result substituted into the second equation implies that $a_2 = a_2 a_3$ **M1**

Continuing $a_2 a_3 - a_2 = 0$, $a_2(a_3 - 1) = 0$ **M1** so either $a_2 = 0$ and thus $a_3 = 0$ in which case the equations are consistent for any a_1 **M1** or $a_3 = 1$ and thus $a_2 = -1$ and $a_1 = -1$ from the third equation. **M1**

Alternatively, from the third equation, $a_3 = a_1 a_2 a_3$, $a_1 a_2 a_3 - a_3 = 0$, $a_3(a_1 a_2 - 1) = 0$

so $a_3 = 0$, $a_2 = 0$ and consistent for any a_1 or $a_1 a_2 = 1$ In the latter case it is simpler to use equation two route again.

Yielding $x^3 - a_1 x^2$ **A1** or $x^3 + x^2 - x - 1$ **A1**

Alternative approach $a_1{}^3 - a_1 a_1{}^2 + a_2 a_1 - a_3 = 0$ (A), $a_2{}^3 - a_1 a_2{}^2 + a_2 a_2 - a_3 = 0$ (B),and

$a_3{}^3 - a_1 a_3{}^2 + a_2 a_3 - a_3 = 0$ (C) **M1**

So (A) implies $a_3 = a_1 a_2$ and thus (B) becomes $a_2{}^3 - a_1 a_2{}^2 + a_2{}^2 - a_1 a_2 = 0$

Hence $a_2(a_2{}^2 - a_1 a_2 + a_2 - a_1) = a_2(a_2 + 1)(a_2 - a_1) = 0$ **M1**

Therefore, $a_2 = 0$ and so from (A) $a_3 = 0$ giving the polynomial $x^3 - a_1 x^2$, which is reflexive **A1**

or $a_2 = -1$ and so from (A) $a_3 = -a_1$ giving the polynomial

$x^3 - a_1x^2 \; - x + \; a_1 = (x - \; a_1)(x^2 - 1)$ for which the equation has roots a_1 and ± 1 . Thus $a_3 = -a_1 = 1$ so the polynomial is $x^3 + x^2 \; - x - 1$ for which the equation has roots -1 , -1 and 1 and so is reflexive **M1 A1**

or $a_2 = a_1$ and so A gives $a_3 = {a_1}^2$ But C is ${a_3}^3 - a_1{a_3}^2 \; + \; a_2a_3 \; - a_3 = 0$ and so

$a_3({a_3}^2 - a_1a_3 \; + \; a_1 \; - a_3) = 0$, $a_3(a_3 \; - a_1)(a_3 - 1) = 0$ **M1**

$a_3 = 0$, $a_1 = 0$, $a_2 = 0$ giving the reflexive polynomial x^3

$a_3 = a_1$, implies $a_1 = 0$ or $a_1 = 1$ giving x^3 again or $a_1 = a_2 = a_3 = 1$ giving the polynomial $x^3 - x^2 \; + \; x \; - 1 = (\, x^2 + 1)(\, x \; - 1)$ which is not reflexive. **E1**

$a_3 = 1$, implies $a_1 = a_2 = \pm 1$ giving 1,1,1 which is not possible or -1,-1,1 (both already considered) **E1** **[11]**

Summary degree 1 $x - a_1$, degree 2 $x^2 - a_1x \;\; (+ \;\, 0)$, and degree 3 $x^3 - a_1x^2$ or

` $x^3 + x^2 \; - x - 1$

(ii)

$$a_1 = \sum_{r=1}^{n} a_r$$

and so,

$$\sum_{r=2}^{n} a_r = 0$$

B1

$$a_2 = \frac{1}{2}\sum_{i \neq j} a_i a_j$$

$$\left(\sum_{r=2}^{n} a_r\right)^2 = \sum_{r=2}^{n} {a_r}^2 + \sum_{i \neq j} a_i a_j - 2a_1 \sum_{r=2}^{n} a_r$$

M1

Thus

$$0 = \sum_{r=2}^{n} {a_r}^2 + 2a_2$$

Hence

$$2a_2 = -{a_2}^2 - \; {a_3}^2 - \; \cdots - {a_n}^2$$

as required. **A1***

$$ {a_2}^2 + 2a_2 + 1 = 1 - \; {a_3}^2 - \; \cdots - {a_n}^2$$

Thus,

$$1-\ a_3{}^2-\ \cdots-a_n{}^2\geq 0$$

So

$$a_3{}^2+\ \cdots+a_n{}^2\leq 1$$

M1

If all the coefficients are integers, then as $a_n\neq 0$, $a_n{}^2=1$ and the other coefficients for $r=3,\dots,n-1$ are zero. But

$$a_n=\prod_{r=1}^{n}a_r$$

so we have established a contradiction. Thus if $a_n\neq 0\,,n\leq 3$ **E1** **[5]**

(iii) So apart from those found in (i) with a_1 integer, any other reflexive polynomials must have $a_n=0$ **E1**

So

$$x^n-a_1x^{n-1}+a_2x^{n-2}-\cdots+(-1)^{n-1}a_{n-1}x=0$$

$$\therefore\ x(x^{n-1}-a_1x^{n-2}+a_2x^{n-3}-\cdots+(-1)^{n-1}a_{n-1})=0$$

M1

which gives a root of zero plus the roots of the bracketed expression. Thus we require the bracketed expression to be itself a reflexive polynomial. This can only happen if either the bracketed expression is of degree 3 or, in turn, $a_{n-1}=0$, and so on. **E1**

Hence, we have $x-a_1$, $x^2-a_1x\ (+\ 0)$, $x^3-a_1x^2$, (with a_1 integer),` x^3+x^2-x-1 , or these multiplied by x^r.

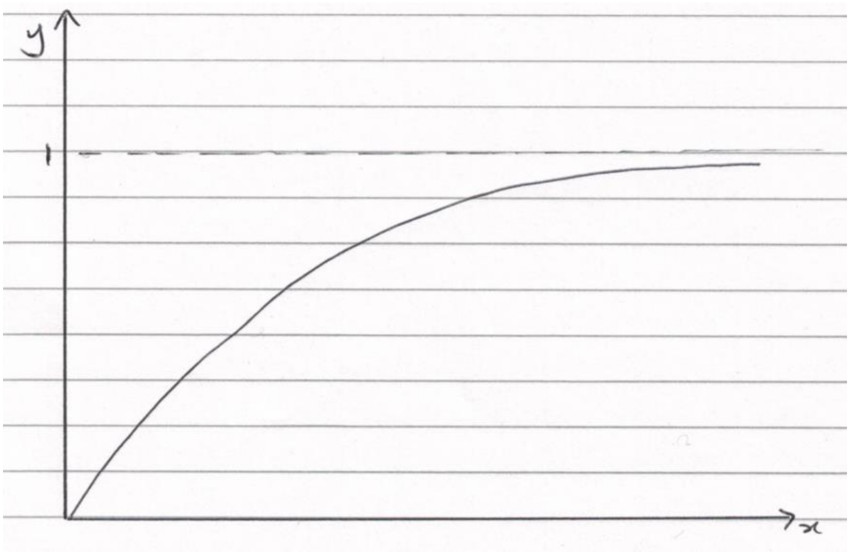
That is $(x-a_1)x^r$ or $(x+1)^2(x-1)x^r$ with $r=0,1,2,\dots$ **A1** **[4]**

5. (i)

$$f(x) = \frac{x}{\sqrt{x^2+p}}$$

$$f'(x) = \frac{\sqrt{x^2+p} - x\frac{1}{2}\frac{2x}{\sqrt{x^2+p}}}{x^2+p} = \frac{p}{(x^2+p)^{\frac{3}{2}}}$$

M1



G1

G1 **[3]**

(ii) for answers using substitution from question paper

$$y = \frac{bx}{\sqrt{x^2+p}} \Rightarrow \frac{dy}{dx} = \frac{bp}{(x^2+p)^{\frac{3}{2}}}$$

$$b^2 - y^2 = b^2 - \frac{b^2x^2}{x^2+p} = \frac{(b^2-b^2)x^2 + b^2p}{x^2+p} = \frac{b^2p}{x^2+p}$$

M1

$$c^2 - y^2 = c^2 - \frac{b^2x^2}{x^2+p} = \frac{(c^2-b^2)x^2 + c^2p}{x^2+p}$$

M1

So

$$\int \frac{1}{(b^2-y^2)\sqrt{c^2-y^2}}\, dy = \int \frac{(x^2+p)}{b^2p} \frac{\sqrt{x^2+p}}{\sqrt{[(c^2-b^2)x^2+c^2p]}} \frac{bp}{(x^2+p)^{\frac{3}{2}}}\, dx$$

M1

$$= \int \frac{1}{b\sqrt{[(c^2-b^2)x^2+c^2p]}}\, dx$$

M1 [4]

Let $c^2 = 2$ M1

Let $b^2 = 3$ M1

Thus

$$\int_1^{\sqrt{2}} \frac{1}{(3-y^2)\sqrt{2-y^2}}\, dy = \int_?^? \frac{1}{3+x^2}\, dx = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_?^?$$

M1 M1 M1 [5]

Let $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$ M1

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2-1)\sqrt{2y^2-1}}\, dy = \int_{\sqrt{2}}^{1} \frac{1}{x}\,\frac{1}{\frac{3}{x^2}-1}\,\frac{1}{\sqrt{\frac{2}{x^2}-1}} \times -\frac{1}{x^2}\, dx = \int_1^{\sqrt{2}} \frac{1}{(3-x^2)\sqrt{2-x^2}}\, dx$$

M1

and so is same answer as previous part A1 ft [3]

(ii) for answers using correct substitution where candidates have realised there is a misprint

$$y = \frac{cx}{\sqrt{x^2+p}} \Rightarrow \frac{dy}{dx} = \frac{cp}{(x^2+p)^{\frac{3}{2}}}$$

$$b^2 - y^2 = b^2 - \frac{c^2x^2}{x^2+p} = \frac{(b^2-c^2)x^2 + b^2p}{x^2+p}$$

$$c^2 - y^2 = c^2 - \frac{c^2x^2}{x^2+p} = \frac{c^2p}{x^2+p}$$

M1

Choose $p = 1$ B1

So

$$\int \frac{1}{(b^2-y^2)\sqrt{c^2-y^2}}\, dy = \int \frac{x^2+1}{b^2+(b^2-c^2)x^2}\,\frac{\sqrt{x^2+1}}{c}\,\frac{c}{(x^2+1)^{\frac{3}{2}}}\, dx$$

M1

$$= \int \frac{1}{b^2+(b^2-c^2)x^2}\, dx$$

A1* [4]

Let $c^2 = 2$ Then $(x^2+1)y^2 = 2x^2$ so $x^2 = \frac{y^2}{2-y^2}$ and when $y = 1, x = 1$

and $y \to \sqrt{2}, x \to \infty$

Let $b^2 = 3$ **B1 M1 A1**

Thus

$$\int_1^{\sqrt{2}} \frac{1}{(3-y^2)\sqrt{2-y^2}}\,dy = \int_1^{\infty} \frac{1}{3+x^2}\,dx = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^{\infty} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

M1 A1 **{5]**

Let $y = \frac{1}{x}, \frac{dy}{dx} = -\frac{1}{x^2}$ **M1**

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2-1)\sqrt{2y^2-1}}\,dy = \int_{\sqrt{2}}^{1} \frac{1}{x}\,\frac{1}{\frac{3}{x^2}-1}\,\frac{1}{\sqrt{\frac{2}{x^2}-1}} \times -\frac{1}{x^2}\,dx = \int_1^{\sqrt{2}} \frac{1}{(3-x^2)\sqrt{2-x^2}}\,dx$$

M1

and so is $\frac{\pi}{3\sqrt{3}}$ **A1 ft** **[3]**

(iii) If

$$y = \frac{bx}{\sqrt{x^2+p}} \Rightarrow \frac{dy}{dx} = \frac{bp}{(x^2+p)^{\frac{3}{2}}}$$

and so

$(x^2+p)y^2 = b^2x^2,\ x^2 = \frac{py^2}{b^2-y^2}$ **M1**

thus

$$\int \frac{1}{(3y^2-1)\sqrt{2y^2-1}}\,dy = \int \frac{1}{\left(\frac{3b^2x^2}{x^2+p}-1\right)\sqrt{\frac{2b^2x^2}{x^2+p}-1}}\,\frac{bp}{(x^2+p)^{\frac{3}{2}}}\,dx$$

$$= \int \frac{bp}{\left(3b^2x^2-(x^2+p)\right)}\,\frac{1}{\sqrt{2b^2x^2-(x^2+p)}}\,dx$$

M1

Choosing $2b^2 = 1$ and $p = -1$ we have **B1**

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{(3y^2-1)\sqrt{2y^2-1}}\,dy = \int_{\infty}^{\sqrt{2}} \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{2}x^2+1}\,dx = \sqrt{2}\int_{\sqrt{2}}^{\infty} \frac{1}{x^2+2}\,dx$$

M1

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{\sqrt{2}}^{\infty} = \frac{\pi}{4}$$

A1 [5]

6. $|z-a|^2 = (z-a)(z-a)^* = (z-a)(z^*-a^*) = zz^* - az^* - a^*z + aa^*$

M1

So z satisfies $|z-a|^2 = r^2$ which means that the locus of P is a circle (C) centre a and radius r (which does not pass through the origin as $r^2 \neq aa^*$.) **A1** **[2]**

(i) As $w = \frac{1}{z}$, $z = \frac{1}{w}$ so

$$\frac{1}{w}\frac{1}{w^*} - a\frac{1}{w^*} - a^*\frac{1}{w} + aa^* - r^2 = 0$$

M1

$$(aa^* - r^2)ww^* - aw - a^*w^* + 1 = 0$$

$$ww^* - \frac{a}{aa^*-r^2}w - \frac{a^*}{aa^*-r^2}w^* + \frac{1}{aa^*-r^2} = 0$$

$$ww^* - \left(\frac{a^*}{aa^*-r^2}\right)^* w - \left(\frac{a^*}{aa^*-r^2}\right)w^* + \left(\frac{a^*}{aa^*-r^2}\right)\left(\frac{a}{aa^*-r^2}\right)$$
$$= \left(\frac{a^*}{aa^*-r^2}\right)\left(\frac{a}{aa^*-r^2}\right) - \frac{1}{aa^*-r^2}$$

$$\left|w - \frac{a^*}{aa^*-r^2}\right|^2 = \frac{aa^*}{(aa^*-r^2)^2} - \frac{1}{aa^*-r^2} = \frac{r^2}{(aa^*-r^2)^2}$$

M1

So C' is a circle centre $\frac{a^*}{aa^*-r^2}$ with radius $\left|\frac{r}{aa^*-r^2}\right|$ **A1** **[3]**

If C and C' are the same circle, then

$$a = \frac{a^*}{aa^*-r^2}$$

M1

and

$$r^2 = \left|\frac{r}{aa^*-r^2}\right|^2$$

M1

Thus

$$r^2(|a|^2 - r^2)^2 = r^2$$

and dividing by $r^2 \neq 0$, $(|a|^2 - r^2)^2 = 1$ as required. **A1*** **[3]**

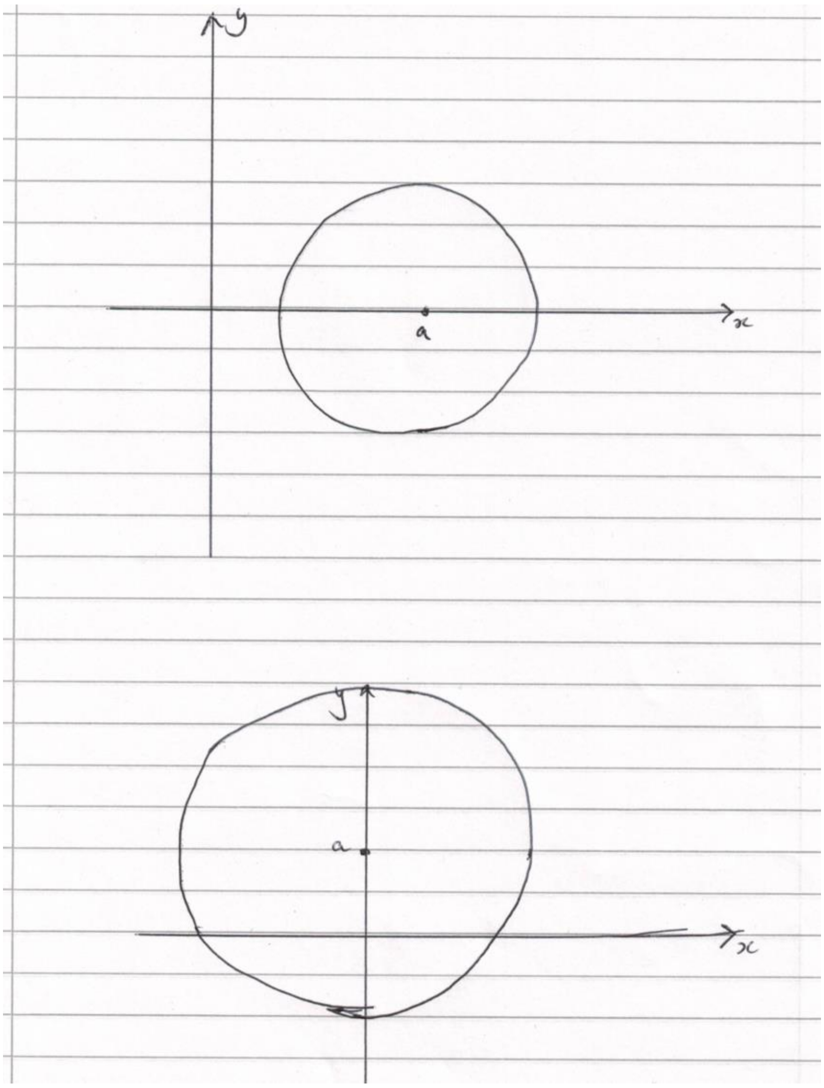
So $|a|^2 - r^2 = \pm 1$ and the equation equating the centres becomes $a = \pm a^*$ **M1**

If $a = c + di$, and $a = a^*$, $c + di = c - di$ implying $d = 0$ and hence a is real. **M1 A1**

G1

If $a = c + di$, and $a = -a^*$, $c + di = -c + di$ implying $c = 0$ and hence a is imaginary. **M1 A1**

G1 **[7]**



(ii) If $w = \frac{1}{z^*}$ then

$$\left|w^* - \frac{a^*}{aa^* - r^2}\right|^2 = \frac{r^2}{(aa^* - r^2)^2}$$

M1

so

$$\left|w - \frac{a}{aa^* - r^2}\right|^2 = \frac{r^2}{(aa^* - r^2)^2}$$

M1

and thus, as before $|a|^2 - r^2 = \pm 1$ but now $a = \frac{a}{aa^* - r^2}$ **A1**

So in the case $|a|^2 - r^2 = +1$, then any a with $|a| = \sqrt{r^2 + 1}$ is possible; in the case $|a|^2 - r^2 = -1$, $a = 0$ (in which case $r = 1$) **A1**

So, it is not the case that a is either real or imaginary. **A1** **[5]**

7. (i)

$$y^2(y^2 - a^2) = x^2(x^2 - a^2)$$

$$y^4 - a^2y^2 + \frac{a^4}{4} = x^4 - a^2x^2 + \frac{a^4}{4}$$

M1

$$\left(y^2 - \frac{a^2}{2}\right)^2 = \left(x^2 - \frac{a^2}{2}\right)^2$$

$$y^2 - \frac{a^2}{2} = \pm\left(x^2 - \frac{a^2}{2}\right)$$

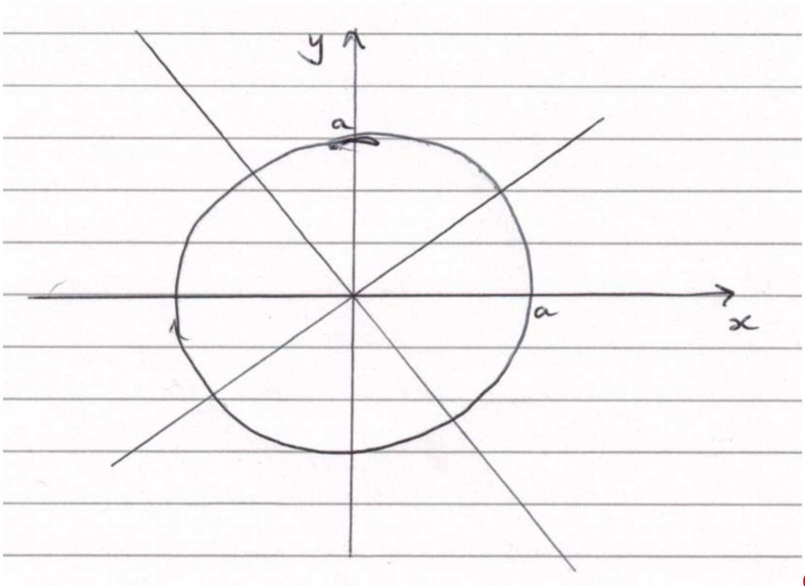
So $x^2 = y^2$ which gives $y = \pm x$

or $x^2 + y^2 = a^2$ **A1**

Alternatively

$$y^4 - x^4 = a^2y^2 - a^2x^2$$

so $(y^2 - x^2)(y^2 + x^2) - a^2(y^2 - x^2) = 0$ **M1**



G1

[3]

(ii)

$$y^2(y^2 - 5) = x^2(x^2 - 4)$$

a)

$$(x^2)^2 - 4x^2 - y^2(y^2 - 5) = 0$$

So for real x^2,

$$16 + 4y^2(y^2 - 5) \geq 0$$

M1

$$(y^2)^2 - 5y^2 + 4 \geq 0$$

$$(y^2-1)(y^2-4) \geq 0$$

$$(y-1)(y+1)(y-2)(y+2) \geq 0$$

M1

As $y \geq 0$,

$$(y-1)(y-2) \geq 0$$

So $0 \leq y \leq 1$ or $y \geq 2$ A1* [3]

b) For small x and y,

$y^4 - 5y^2 = x^4 - 4x^2$ becomes $5y^2 \approx 4x^2$ so $y \approx \pm\frac{2x}{\sqrt{5}}$ B1

For large x and y,

$y^4 - 5y^2 = x^4 - 4x^2$ becomes $y^4 \approx x^4$ so $y \approx \pm x$ B1 [2]

c)

$$y^2(y^2-5) = x^2(x^2-4)$$

$$(4y^3 - 10y)\frac{dy}{dx} = 4x^3 - 8x$$

M1

$$\frac{dy}{dx} = 0 \;\Rightarrow\; 4x^3 - 8x = 0$$

$$4x(x^2-2) = 0$$

M1

So $x = 0$, $y = 0, \sqrt{5}$ or $x = \sqrt{2}$, $y = 1, 2$

Thus $(0,\sqrt{5})$, $(\sqrt{2},1)$, $(\sqrt{2},2)$ but not $(0,0)$ A1 B1

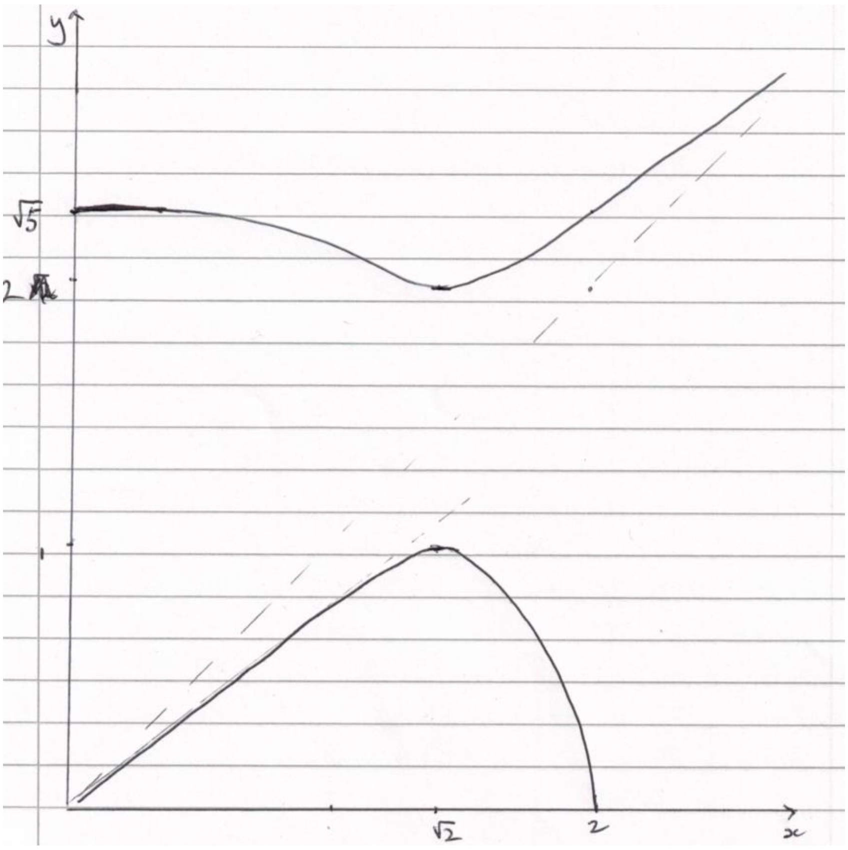
$$\frac{dx}{dy} = 0 \;\Rightarrow\; 4y^3 - 10y = 0$$

$$2y(2y^2-5) = 0$$

M1

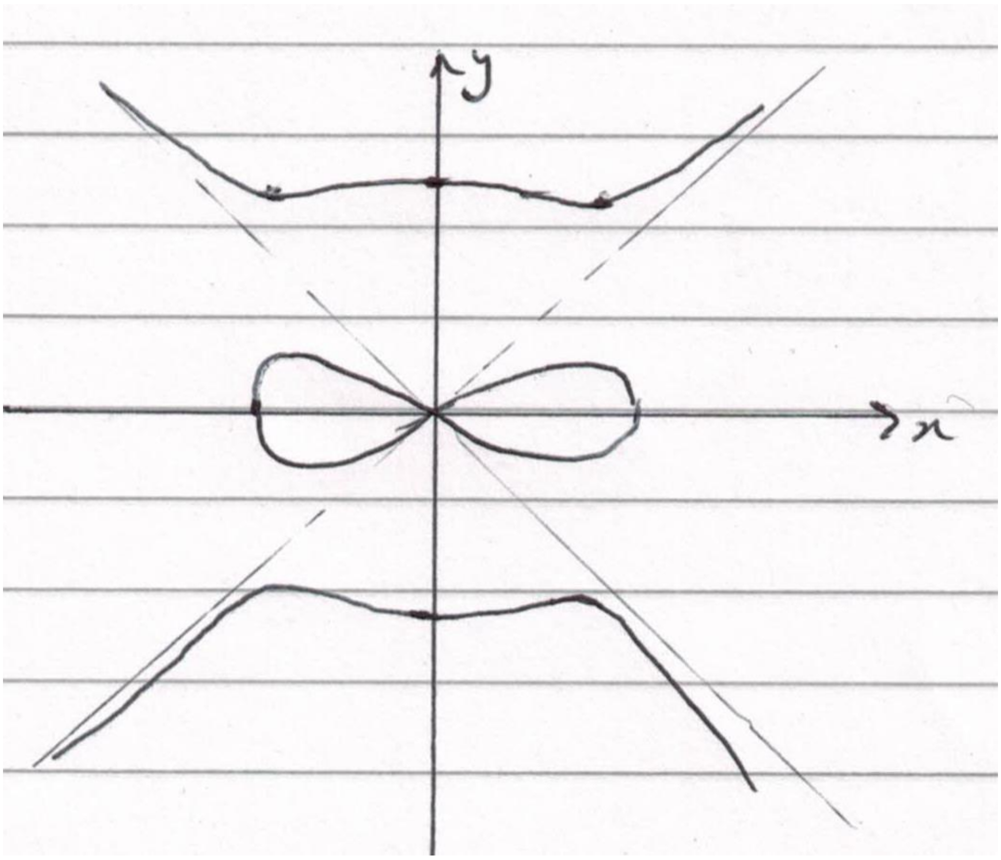
So $y = 0$, $x = 0, 2$ but $y = \sqrt{\frac{5}{2}}$ gives x complex E1

Thus $(2,0)$ but not $(0,0)$ A1 [7]



G1 G1 G1 G1 **[4]**

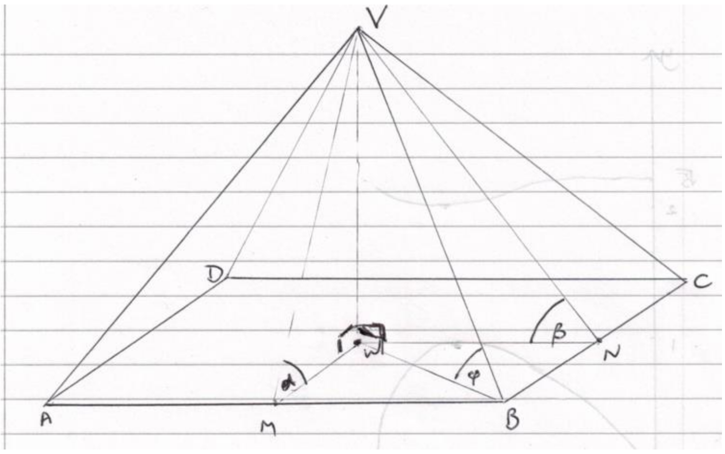
(iii) **G1 ft** **[1]**



8. (i) If W is centre of base, and M is midpoint of AB, then

$$\overrightarrow{MV} = \lambda \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}$$

M1



So a unit vector perpendicular to AVB is $\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix}$ **A1 (or B2)**

Similarly, a unit vector perpendicular to BVC is $\begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix}$ **B1**

AS the obtuse angle between AVB and BVC is $\pi - \theta$, the acute angle between the two unit vectors is θ , and so

$$\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} = \cos\theta$$

M1

Hence $\cos\theta = \cos\alpha\cos\beta$ as required. **A1*** **[5]**

(ii) $MW = VW\cot\alpha$, $BM = NW = VW\cot\beta$, $BW = VW\cot\varphi$ **M1**

By Pythagoras, $MW^2 + BW^2 = BW^2$ and so $\cot^2\alpha + \cot^2\beta = \cot^2\varphi$ **M1**

So

$$\tan^2\varphi = \frac{1}{\cot^2\alpha + \cot^2\beta}$$

and thus

$$\sec^2\varphi = \frac{1}{\cot^2\alpha + \cot^2\beta} + 1$$

M1

giving

$$\cos^2\varphi = \frac{\cot^2\alpha + \cot^2\beta}{\cot^2\alpha + \cot^2\beta + 1} = \frac{\tan^2\alpha + \tan^2\beta}{\tan^2\alpha + \tan^2\beta + \tan^2\alpha\tan^2\beta}$$

M1

$$= \frac{\sec^2\alpha + \sec^2\beta - 2}{\sec^2\alpha + \sec^2\beta - 2 + (\sec^2\alpha - 1)(\sec^2\beta - 1)}$$

M1

$$= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\alpha\cos^2\beta}{\cos^2\alpha + \cos^2\beta - 2\cos^2\alpha\cos^2\beta + (1 - \cos^2\alpha)(1 - \cos^2\beta)}$$

M1

$$= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\alpha\cos^2\beta}$$

$$= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\theta}$$

M1 A1* [8]

$$(\cos\alpha - \cos\beta)^2 \geq 0$$

Thus

$$\cos^2\alpha + \cos^2\beta \geq 2\cos\alpha\cos\beta = 2\cos\theta$$

M1

So

$$\cos^2\varphi = \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\theta} \geq \frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta}$$

A1* **[2]**

$$\frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta} = \frac{2\cos\theta\,(1 - \cos\theta)}{(1 - \cos\theta)(1 + \cos\theta)}$$

$1 - \cos\theta \neq 0$ as θ is acute. **E1**

Thus

$$\cos^2\varphi \geq \frac{2\cos\theta}{(1 + \cos\theta)} = \frac{2}{(1 + \cos\theta)}\cos\theta$$

As θ is acute, $(1 + \cos\theta) < 2$ and so $\frac{2}{(1+\cos\theta)}\cos\theta > \cos\theta$ **E1**

But also $\cos\theta > \cos\theta\cos\theta$ **E1**

Hence, $\cos^2\varphi \geq \cos^2\theta$

As both $\cos\varphi$ and $\cos\theta$ are positive $\cos\varphi \geq \cos\theta$, and so $\varphi \leq \theta$ **E1 E1 [5]**

9. (i)

$$\boldsymbol{r} = (a\sin\theta - s)\boldsymbol{i} + a\cos\theta\,\boldsymbol{j}$$

B1

So differentiating with respect to time,

$$\dot{\boldsymbol{r}} = \left(a\dot{\theta}\cos\theta - \dot{s}\right)\boldsymbol{i} - a\dot{\theta}\sin\theta\ \boldsymbol{j}$$

M1 **[2]**

Conserving linear momentum horizontally, **M1**

$$m\left(a\dot{\theta}\cos\theta - \dot{s}\right) - M\dot{s} = 0$$

A1

Thus $ma\dot{\theta}\cos\theta = (m+M)\dot{s}$, and so $\dot{s} = \frac{m}{m+M}a\dot{\theta}\cos\theta = \left(1 - \frac{M}{m+M}\right)a\dot{\theta}\cos\theta$

$$\dot{s} = (1-k)a\dot{\theta}\cos\theta$$

M1 **[3]**

Hence,

$$\dot{\boldsymbol{r}} = \left(a\dot{\theta}\cos\theta - (1-k)a\dot{\theta}\cos\theta\right)\boldsymbol{i} - a\dot{\theta}\sin\theta\ \boldsymbol{j}$$

$$= a\dot{\theta}(k\cos\theta\,\boldsymbol{i} - \sin\theta\ \boldsymbol{j})$$

M1 **[1]**

(ii) Conserving energy, **M1**

$$mga = mga\cos\theta + \frac{1}{2}M\left[(1-k)a\dot{\theta}\cos\theta\right]^2 + \frac{1}{2}m\left(a\dot{\theta}\right)^2[(k\cos\theta)^2 + \sin^2\theta]$$

A1

So

$$2mg(1-\cos\theta) = a\dot{\theta}^2(M(1-k)^2\cos^2\theta + mk^2\cos^2\theta + m\sin^2\theta)$$

That is

$$2g(1-\cos\theta) = a\dot{\theta}^2\left(\frac{M}{m}(1-k)^2\cos^2\theta + k^2\cos^2\theta + \sin^2\theta\right)$$

As

$$k = \frac{M}{m+M}$$

$$\frac{m+M}{M} = \frac{1}{k}$$

$$\frac{m}{M} = \frac{1}{k} - 1 = \frac{1-k}{k}$$

and so **M1**

$$\frac{M}{m} = \frac{k}{1-k}$$

$$2g(1-\cos\theta) = a\dot{\theta}^2\left(\frac{k}{1-k}(1-k)^2\cos^2\theta + k^2\cos^2\theta + \sin^2\theta\right)$$

M1

$$= a\dot{\theta}^2(k(1-k)\cos^2\theta + k^2\cos^2\theta + \sin^2\theta)$$

$$= a\dot{\theta}^2(k\cos^2\theta + \sin^2\theta)$$

A1* **[5]**

(iii)

$$\ddot{\boldsymbol{r}} = a\ddot{\theta}(k\cos\theta\,\boldsymbol{i} - \sin\theta\;\boldsymbol{j}) + a\dot{\theta}\left(-k\dot{\theta}\sin\theta\,\boldsymbol{i} - \dot{\theta}\cos\theta\;\boldsymbol{j}\right)$$

M1

$$= a\left[k\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\boldsymbol{i} - \left(\ddot{\theta}\;\sin\theta + \dot{\theta}^2\cos\theta\right)\boldsymbol{j}\right]$$

A1

When the particle loses contact with the sphere, $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$ **E1**

So, $\ddot{\boldsymbol{r}}.(\sin\theta\,\boldsymbol{i} + k\cos\theta\,\boldsymbol{j}) = -g\boldsymbol{j}\,.(\sin\theta\,\boldsymbol{i} + k\cos\theta\,\boldsymbol{j})$ with $\theta = \alpha$ **M1**

Therefore, $a\left[k\ddot{\theta}\cos\alpha\sin\alpha\; - k\dot{\theta}^2\sin^2\alpha - k\ddot{\theta}\cos\alpha\sin\alpha - k\,\dot{\theta}^2\cos^2\alpha\right] = -kg\cos\alpha$ **M1**

which simplifies to $a\dot{\theta}^2 = g\cos\alpha$ **A1*** **[6]**

Substituting in the final result of (ii),

$$g\cos\alpha\;\;(k\cos^2\alpha + \sin^2\alpha) = 2g(1-\cos\alpha)$$

Thus **M1**

$$(k-1)\cos^3\alpha + 3\cos\alpha - 2 = 0$$

A1

$$3\cos\alpha - 2 = (1-k)\cos^3\alpha$$

$k < 1$, and $\cos\alpha > 0$ so $3\cos\alpha - 2 > 0$ and so $\cos\alpha > \frac{2}{3}$ **E1** **[3]**

10. (i) As the spheres are smooth, as is the table, the momentum of P perpendicular to the direction of the line of centres is unchanged so

$$mu \sin\alpha = mv \sin(\alpha + \theta)$$

M1

and so

$$u \sin\alpha = v \sin(\alpha + \theta)$$

as required. **A1*** **[2]**

Conserving momentum in the direction of the line of centres,

$$mu \cos\alpha = mv \cos(\alpha + \theta) + mw$$

M1 A1

and so

$$u \cos\alpha = v \cos(\alpha + \theta) + w$$

Eliminating u,

$$v \sin(\alpha + \theta) \cos\alpha = v \cos(\alpha + \theta) \sin\alpha + w \sin\alpha$$

M1

So

$$v \sin(\alpha + \theta) \cos\alpha - v \cos(\alpha + \theta) \sin\alpha = w \sin\alpha$$

That is

$$v \sin\theta = w \sin\alpha$$

$$w = v \frac{\sin\theta}{\sin\alpha}$$

M1 A1 **[5]**

[Alternatively, this result can be obtained by conserving momentum perpendicular to the original direction of motion of P.

$$0 = mw \sin\alpha - mv \sin\theta$$ **M2 A1**

(ii) Newton's experimental law of impact in the direction of the line of centres gives

$$w - v \cos(\alpha + \theta) = eu \cos\alpha$$

M1 A1

Substituting for w and u in terms of v gives

$$v \frac{\sin\theta}{\sin\alpha} - v \cos(\alpha + \theta) = \frac{ev \sin(\alpha + \theta) \cos\alpha}{\sin\alpha}$$

M1

Thus

$$\sin\theta - \cos(\alpha + \theta) \sin\alpha = e \sin(\alpha + \theta) \cos\alpha$$

or that is

$$\sin\theta = \cos(\alpha+\theta)\sin\alpha + e\sin(\alpha+\theta)\cos\alpha$$

A1* **[4]**

Expanding and dividing through by $\cos\theta$

$$\tan\theta = \cos\alpha\sin\alpha - \sin^2\alpha\tan\theta + e\sin\alpha\cos\alpha + e\cos^2\alpha\tan\theta$$

M1

$$\tan\theta\,(1+\sin^2\alpha - e\cos^2\alpha) = (1+e)\sin\alpha\cos\alpha$$

Thus, dividing by $\cos^2\alpha$ **M1**

$$\tan\theta\,(\sec^2\alpha + \tan^2\alpha - e) = (1+e)\tan\alpha$$

$$\tan\theta = \frac{(1+e)\tan\alpha}{1+2\tan^2\alpha - e}$$

A1 **[3]**

Let $t = \tan\alpha$,

$$\tan\theta = \frac{(1+e)t}{1-e+2t^2}$$

$$\frac{d(\tan\theta)}{dt} = \frac{(1-e+2t^2)(1+e) - 4t(1+e)t}{(1-e+2t^2)^2}$$

M1 A1

For a maximum,

$$\frac{d(\tan\theta)}{dt} = 0$$

$$(1-e+2t^2) - 4t^2 = 0$$

M1

$$t = \sqrt{\frac{1-e}{2}}$$

A1

$$\tan\theta = \frac{(1+e)}{2(1-e)}\sqrt{\frac{1-e}{2}} = \frac{\sqrt{2}(1+e)}{4\sqrt{1-e}}$$

A1 ft

This is the only stationary value and must be a maximum as for $t=0$, $\tan\theta = 0$ and as $t\to\infty$, $\tan\theta\to 0$, and for all t , $\tan\theta > 0$. **E1** **[6]**

11. (i) If X is the number of customers who take sand,

$$P(X = r) = \sum_{i=r}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \times \binom{i}{r} p^r (1-p)^{i-r}$$

M1 A1 A1

$$= \frac{(\lambda p)^r e^{-\lambda}}{r!} \sum_{I=0}^{\infty} \frac{[\lambda(1-p)]^i}{i!}$$

M1

$$= \frac{(\lambda p)^r e^{-\lambda}}{r!} e^{\lambda(1-p)}$$

$$= \frac{(\lambda p)^r e^{-\lambda p}}{r!}$$

A1*

and so it follows a Poisson distribution with mean λp . **[5]**

(ii)

$$E(mass\ taken)$$

$$= 0.e^{-\lambda p} + kS.\frac{(\lambda p)^1 e^{-\lambda p}}{1!} + [kS + k(1-k)S].\frac{(\lambda p)^2 e^{-\lambda p}}{2!}$$

$$+ [kS + k(1-k)S + k(1-k)^2 S].\frac{(\lambda p)^3 e^{-\lambda p}}{3!} + \cdots$$

M1 A1

$$= e^{-\lambda p} kS \left[\lambda p + \frac{1-(1-k)^2}{1-(1-k)} \frac{(\lambda p)^2}{2!} + \frac{1-(1-k)^3}{1-(1-k)} \frac{(\lambda p)^3}{3!} + \cdots\right]$$

for $k \neq 0$ **E1**

$$= e^{-\lambda p} \frac{kS}{k} \left[\left(\lambda p + \frac{(\lambda p)^2}{2!} + \frac{(\lambda p)^3}{3!} + \cdots\right) - \left((1-k)\lambda p + \frac{(1-k)^2(\lambda p)^2}{2!} + \frac{(1-k)^3(\lambda p)^3}{3!} + \cdots\right)\right]$$

$$= e^{-\lambda p} S\left((e^{\lambda p} - 1) - (e^{(1-k)\lambda p} - 1)\right)$$

M1

$$= S(1 - e^{-k\lambda p})$$

A1*

If $k = 0$, then the expected sand taken is zero which is $S(1 - e^{-k\lambda p})$ **E1** **[6]**

(iii) Using the working form part (i), the probability that r customers take sand is

$$\frac{(\lambda p)^r e^{-\lambda p}}{r!}$$

and from (ii) the mass of sand taken is

$$kS\frac{1-(1-k)^r}{1-(1-k)} = S(1-(1-k)^r)$$

so the amount that the merchant's assistant takes is $S(1-k)^r$.

The probability that the assistant takes the golden grain in this case is thus

$$\frac{kS(1-k)^r}{S} = k(1-k)^r$$

M1 A1

The probability that the assistant takes the golden grain is

$$P = \sum_{r=0}^{\infty} k(1-k)^r \frac{(\lambda p)^r e^{-\lambda p}}{r!} = ke^{-\lambda p}\sum_{r=0}^{\infty}\frac{[(1-k)\lambda p]^r}{r!} = ke^{-\lambda p}e^{(1-k)\lambda p} = ke^{-k\lambda p}$$

M1 A1 **[4]**

If $k=0$, no sand is taken by anyone including the assistant so P should be zero, $0e^0 = 0$ **E1**

As $k \to 1$, $ke^{-k\lambda p} \to e^{-\lambda p}$ which is the probability that no customer takes sand, in the limit, the only way the assistant can take sand, in which case the grain is bound to be in his sand.. **E1** **[2]**

$$P = ke^{-k\lambda p}$$

$$\frac{dP}{dk} = e^{-k\lambda p} - k\lambda p e^{-k\lambda p} = (1-k\lambda p)e^{-k\lambda p}$$

M1

For a maximum, $\frac{dP}{dk} = 0$, so $k = \frac{1}{\lambda p}$. (Note $p\lambda > 1$, so $\frac{1}{\lambda p} < 1$ as required because $k < 1$) **A1**

Justifying that this gives a maximum either:-

$k < \frac{1}{\lambda p} \Rightarrow \frac{dP}{dk} > 0$ and $k > \frac{1}{\lambda p} \Rightarrow \frac{dP}{dk} < 0$ as required

or

$$\frac{d^2P}{dk^2} = -\lambda p(1-k\lambda p)e^{-k\lambda p} - \lambda p e^{-k\lambda p} = \lambda p e^{-k\lambda p}[k\lambda p - 2]$$

and for $k = \frac{1}{\lambda p}, \frac{d^2P}{dk^2} = -\lambda p e^{-1} < 0$ as required. **E1** **[3]**

12. For each subset, each integer can be in it or not. Hence the number of possibilities is
$2 \times 2 \times 2 \ldots \times 2 = 2^n$. **E1** **[1]**

(i) $P(1 \in A_1) = \frac{1}{2}$ **B1** **[1]**

(ii) $P(t \in A_1 \cap A_2) = \frac{1}{4}$ **B1**

so $P(t \notin A_1 \cap A_2) = \frac{3}{4}$ **M1**

For $A_1 \cap A_2 = \emptyset$, no element is in the intersection, so $P(\, A_1 \cap A_2 = \emptyset\,) = \left(\frac{3}{4}\right)^n$ **M1**

$$P(\, A_1 \cap A_2 \cap A_3 = \emptyset\,) = \left(\frac{7}{8}\right)^n$$

M1 A1 (or B2)

$$P(\, A_1 \cap A_2 \cap \ldots \cap A_m = \emptyset\,) = \left(1 - \frac{1}{2^m}\right)^n$$

M1 A1 (or B2) [7]

(iii) $A_1 \subseteq A_2 \Rightarrow t \in A_1 \cap A_2\,, t \in A_1' \cap A_2'\,, or,\; t \in A_1' \cap A_2$

M1

So $P(A_1 \subseteq A_2) = \left(\frac{3}{4}\right)^n$ **M1A1** **[3]**

$A_1 \subseteq A_2 \subseteq A_3 \Rightarrow t \in A_1 \cap A_2 \cap A_3\,, t \in {A_1}' \cap A_2 \cap A_3, t \in {A_1}' \cap {A_2}' \cap A_3 \; or\,, t \in {A_1}' \cap {A_2}' \cap {A_3}'$

M1A1

So $P(A_1 \subseteq A_2 \subseteq A_3) = \left(\frac{4}{8}\right)^n = \left(\frac{1}{2}\right)^n$ **M1A1** **[4]**

$A_1 \subseteq A_2 \ldots A_m \Rightarrow t \in A_1 \cap A_2 \ldots A_m\,, t \in A_1' \cap A_2,,, A_m, t \in A_1' \cap A_2' \ldots A_m \ldots\,, t \in A_1' \cap A_2' \ldots {A_m}'$

M1A1

$$P(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m) = \left(\frac{m+1}{2^m}\right)^n$$

M1A1 **[4]**

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