Cambridge Assessment Admissions Testing

Sixth Term Examination Papers **9470**
MATHEMATICS 2 Morning
Thursday 9 June 2022 Time: 3 hours

* 1 7 3 5 3 5 9 9 9 9 *

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You must shade the appropriate Question Answered circle on every page of the answer booklet that you write on. Failure to do so might mean that some of your answers are not marked.

There is NO Mathematical Formulae Booklet.

Calculators are not permitted.

Wait to be told you may begin before turning this page.

BLANK PAGE

Section A: Pure Mathematics

1 **(i)** By integrating one of the two terms in the integrand by parts, or otherwise, find

$$\int \left(2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}} \right) \mathrm{d}x\,.$$

(ii) Find

$$\int (x^2+2)\frac{\sin x}{x^3}\,\mathrm{d}x\,.$$

(iii) **(a)** Sketch the graph with equation $y = \dfrac{\mathrm{e}^x}{x}$, giving the coordinates of any stationary points.

(b) Find a if

$$\int_a^{2a} \frac{\mathrm{e}^x}{x}\,\mathrm{d}x = \int_a^{2a} \frac{\mathrm{e}^x}{x^2}\,\mathrm{d}x\,.$$

(c) Show that it is not possible to find distinct integers m and n such that

$$\int_m^{n} \frac{\mathrm{e}^x}{x}\,\mathrm{d}x = \int_m^{n} \frac{\mathrm{e}^x}{x^2}\,\mathrm{d}x\,.$$

2 A sequence u_n, where $n = 1, 2, \ldots$, is said to have *degree* d if u_n, as a function of n, is a polynomial of degree d.

(i) Show that, in any sequence u_n $(n = 1, 2, \ldots)$ that satisfies $u_{n+1} = \frac{1}{2}(u_{n+2} + u_n)$ for all $n \geqslant 1$, there is a constant difference between successive terms.

Deduce that any sequence u_n for which $u_{n+1} = \frac{1}{2}(u_{n+2} + u_n)$, for all $n \geqslant 1$, has degree at most 1.

(ii) The sequence v_n $(n = 1, 2, \ldots)$ satisfies $v_{n+1} = \frac{1}{2}(v_{n+2} + v_n) - p$ for all $n \geqslant 1$, where p is a non-zero constant. By writing $v_n = t_n + pn^2$, show that the sequence v_n has degree 2.

Given that $v_1 = v_2 = 0$, find v_n in terms of n and p.

(iii) The sequence w_n $(n = 1, 2, \ldots)$ satisfies $w_{n+1} = \frac{1}{2}(w_{n+2} + w_n) - an - b$ for all $n \geqslant 1$, where a and b are constants with $a \neq 0$. Show that the sequence w_n has degree 3.

Given that $w_1 = w_2 = 0$, find w_n in terms of n, a and b.

3 The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geqslant 0$, $F_{n+2} = F_{n+1} + F_n$.

(i) Prove that $F_r \leqslant 2^{r-n} F_n$ for all $n \geqslant 1$ and all $r \geqslant n$.

(ii) Let $S_n = \displaystyle\sum_{r=1}^{n} \frac{F_r}{10^r}$.

Show that

$$\sum_{r=1}^{n} \frac{F_{r+1}}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{10^{r-1}} = 89S_n - 10F_1 - F_0 + \frac{F_n}{10^n} + \frac{F_{n+1}}{10^{n-1}}\,.$$

(iii) Show that $\displaystyle\sum_{r=1}^{\infty} \frac{F_r}{10^r} = \frac{10}{89}$ and that $\displaystyle\sum_{r=7}^{\infty} \frac{F_r}{10^r} < 2 \times 10^{-6}$. Hence find, with justification, the first six digits after the decimal point in the decimal expansion of $\frac{1}{89}$.

(iv) Find, with justification, a number of the form $\dfrac{r}{s}$ with r and s both positive integers less than 10000 whose decimal expansion starts

$$0.000101020305081321345 5\ldots.$$

4 **(i)** Show that the function f, given by the single formula $\mathrm{f}(x) = \big|x\big| - \big|x-5\big| + 1$, can be written without using modulus signs as

$$\mathrm{f}(x) = \begin{cases} -4 & x \leqslant 0\,, \\ 2x-4 & 0 \leqslant x \leqslant 5\,, \\ 6 & 5 \leqslant x\,. \end{cases}$$

Sketch the graph with equation $y = \mathrm{f}(x)$.

(ii) The function g is given by:

$$\mathrm{g}(x) = \begin{cases} -x & x \leqslant 0\,, \\ 3x & 0 \leqslant x \leqslant 5\,, \\ x+10 & 5 \leqslant x\,. \end{cases}$$

Use modulus signs to write $\mathrm{g}(x)$ as a single formula.

(iii) Sketch the graph with equation $y = \mathrm{h}(x)$, where $\mathrm{h}(x) = x^2 - x - 4\big|x\big| + \big|x(x-5)\big|$.

(iv) The function k is given by:

$$\mathrm{k}(x) = \begin{cases} 10x & x \leqslant 0\,, \\ 2x^2 & 0 \leqslant x \leqslant 5\,, \\ 50 & 5 \leqslant x\,. \end{cases}$$

Use modulus signs to write $\mathrm{k}(x)$ as a single formula, explicitly verifying that your formula is correct.

5 **(i)** Given that $a > b > c > 0$ are constants, and that x, y, z are non-negative variables, show that

$$ax + by + cz \leqslant a(x + y + z).$$

In the acute-angled triangle ABC, a, b and c are the lengths of sides BC, CA and AB, respectively, with $a > b > c$. P is a point inside, or on the sides of, the triangle, and x, y and z are the perpendicular distances from P to BC, CA and AB, respectively. The area of the triangle is Δ.

(ii) **(a)** Find Δ in terms of a, b, c, x, y and z.

(b) Find both the minimum value of the sum of the perpendicular distances from P to the three sides of the triangle and the values of x, y and z which give this minimum sum, expressing your answers in terms of some or all of a, b, c and Δ.

(iii) **(a)** Show that, for all real a, b, c, x, y and z,

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 + (ax + by + cz)^2.$$

(b) Find both the minimum value of the sum of the squares of the perpendicular distances from P to the three sides of the triangle and the values of x, y and z which give this minimum sum, expressing your answers in terms of some or all of a, b, c and Δ.

(iv) Find both the maximum value of the sum of the squares of the perpendicular distances from P to the three sides of the triangle and the values of x, y and z which give this maximum sum, expressing your answers in terms of some or all of a, b, c and Δ.

6 In this question, you should consider only points lying in the first quadrant, that is with $x > 0$ and $y > 0$.

(i) The equation $x^2 + y^2 = 2ax$ defines a *family* of curves in the first quadrant, one curve for each positive value of a. A second family of curves in the first quadrant is defined by the equation $x^2 + y^2 = 2by$, where $b > 0$.

(a) Differentiate the equation $x^2 + y^2 = 2ax$ implicitly with respect to x, and hence show that every curve in the first family satisfies the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2.$$

Find similarly a differential equation, independent of b, for the second family of curves.

(b) Hence, or otherwise, show that, at every point with $y \neq x$ where a curve in the first family meets a curve in the second family, the tangents to the two curves are perpendicular.

A curve in the first family meets a curve in the second family at $(c\,,\ c)$, where $c > 0$. Find the equations of the tangents to the two curves at this point. Is it true that where a curve in the first family meets a curve in the second family on the line $y = x$, the tangents to the two curves are perpendicular?

(ii) Given the family of curves in the first quadrant $y = c\ln x$, where c takes any non-zero value, find, by solving an appropriate differential equation, a second family of curves with the property that at every point where a curve in the first family meets a curve in the second family, the tangents to the two curves are perpendicular.

(iii) A family of curves in the first quadrant is defined by the equation $y^2 = 4k(x + k)$, where k takes any non-zero value.

Show that, at every point where one curve in this family meets a second curve in the family, the tangents to the two curves are perpendicular.

7 Let $\mathrm{h}(z) = nz^6 + z^5 + z + n$, where z is a complex number and $n \geqslant 2$ is an integer.

(i) Let w be a root of the equation $\mathrm{h}(z) = 0$.

(a) Show that $|w^5| = \sqrt{\dfrac{\mathrm{f}(w)}{\mathrm{g}(w)}}$, where

$$\mathrm{f}(z) = n^2 + 2n\mathrm{Re}(z) + |z|^2 \text{ and } \mathrm{g}(z) = n^2|z|^2 + 2n\mathrm{Re}(z) + 1.$$

(b) By considering $\mathrm{f}(w) - \mathrm{g}(w)$, prove by contradiction that $|w| \geqslant 1$.

(c) Show that $|w| = 1$.

(ii) It is given that the equation $\mathrm{h}(z) = 0$ has six distinct roots, none of which is purely real.

(a) Show that $\mathrm{h}(z)$ can be written in the form

$$\mathrm{h}(z) = n(z^2 - a_1 z + 1)(z^2 - a_2 z + 1)(z^2 - a_3 z + 1),$$

where a_1, a_2 and a_3 are real constants.

(b) Find $a_1 + a_2 + a_3$ in terms of n.

(c) By considering the coefficient of z^3 in $\mathrm{h}(z)$, find $a_1 a_2 a_3$ in terms of n.

(d) How many of the six roots of the equation $\mathrm{h}(z) = 0$ have a negative real part? Justify your answer.

8 Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a \neq d$. The transformation represented by $\mathbf{M}$ has exactly two distinct invariant lines through the origin.

(i) Show that, if neither invariant line is the y-axis, then the gradients of the invariant lines are the roots of the equation

$$bm^2 + (a - d)m - c = 0.$$

If one invariant line is the y-axis, what is the gradient of the other?

(ii) Show that, if the angle between the two invariant lines is $45°$, then

$$(a - d)^2 = (b - c)^2 - 4bc.$$

(iii) Find a necessary and sufficient condition, on some or all of a, b, c and d, for the two invariant lines to make equal angles with the line $y = x$.

(iv) Give an example of a matrix which satisfies both the conditions in parts **(ii)** and **(iii)**.

Section B: Mechanics

9 A rectangular prism is fixed on a horizontal surface. A vertical wall, parallel to a vertical face of the prism, stands at a distance d from it. A light plank, making an acute angle θ with the horizontal, rests on an upper edge of the prism and is in contact with the wall below the level of that edge of the prism and above the level of the horizontal plane. You may assume that the plank is long enough and the prism high enough to make this possible.

The contact between the plank and the prism is smooth, and the coefficient of friction at the contact between the plank and the wall is μ. When a heavy point mass is fixed to the plank at a distance x, along the plank, from its point of contact with the wall, the system is in equilibrium.

(i) Show that, if $x = d\sec^3\theta$, then there is no frictional force acting between the plank and the wall.

(ii) Show that, if $x > d\sec^3\theta$, it is necessary that

$$\mu \geqslant \frac{x - d\sec^3\theta}{x\tan\theta}$$

and give the corresponding inequality if $x < d\sec^3\theta$.

(iii) Show that

$$\frac{x}{d} \geqslant \frac{\sec^3\theta}{1+\mu\tan\theta}.$$

Show also that, if $\mu < \cot\theta$, then

$$\frac{x}{d} \leqslant \frac{\sec^3\theta}{1-\mu\tan\theta}.$$

(iv) Show that if x is such that the point mass is fixed to the plank somewhere between the edge of the prism and the wall, then $\tan\theta < \mu$.

10 **(i)** Show that, if a particle is projected at an angle α above the horizontal with speed u, it will reach height h at a horizontal distance s from the point of projection where

$$h = s\tan\alpha - \frac{gs^2}{2u^2\cos^2\alpha}.$$

The remainder of this question uses axes with the x- and y-axes horizontal and the z-axis vertically upwards. The ground is a sloping plane with equation $z = y\tan\theta$ and a road runs along the x-axis. A cannon, which may have any angle of inclination and be pointed in any direction, fires projectiles from ground level with speed u. Initially, the cannon is placed at the origin.

(ii) Let a point P on the plane have coordinates $(x\,,\ y\,,\ y\tan\theta)$. Show that the condition for it to be possible for a projectile from the cannon to land at point P is

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 \leqslant \frac{u^4\sec^2\theta}{g^2}.$$

(iii) Show that the furthest point directly up the plane that can be reached by a projectile from the cannon is a distance

$$\frac{u^2}{g(1+\sin\theta)}$$

from the cannon.

How far from the cannon is the furthest point directly down the plane that can be reached by a projectile from it?

(iv) Find the length of road which can be reached by projectiles from the cannon.

The cannon is now moved to a point on the plane vertically above the y-axis, and a distance r from the road. Find the value of r which maximises the length of road which can be reached by projectiles from the cannon. What is this maximum length?

Section C: Probability and Statistics

11 A batch of N USB sticks is to be used on a network. Each stick has the same unknown probability p of being infected with a virus. Each stick is infected, or not, independently of the others.

The network manager decides on an integer value of T with $0 \leqslant T < N$. If $T = 0$ no testing takes place and the N sticks are used on the network, but if $T > 0$, the batch is subject to the following procedure.

- Each of T sticks, chosen at random from the batch, undergoes a test during which it is destroyed.
- If any of these T sticks is infected, all the remaining $N - T$ sticks are destroyed.
- If none of the T sticks is infected, the remaining $N - T$ sticks are used on the network.

If any stick used on the network is infected, the network has to be disinfected at a cost of £D, where $D > 0$. If no stick used on the network is infected, there is a gain of £1 for each of the $N - T$ sticks. There is no cost to testing or destroying a stick.

(i) Find an expression in terms of N, T, D and q, where $q = 1 - p$, for the expected net loss.

(ii) Let $\alpha = \dfrac{DT}{N(N - T + D)}$. Show that $0 \leqslant \alpha < 1$.

Show that, for fixed values of N, D and T, the greatest value of the expected net loss occurs when q satisfies the equation $q^{N-T} = \alpha$.

Show further that this greatest value is £$\dfrac{D(N - T)\,\alpha^k}{N}$, where $k = \dfrac{T}{N - T}$.

(iii) For fixed values of N and D, show that there is some $\beta > 0$ so that for all $p < \beta$, the expression for the expected loss found in part **(i)** is an increasing function of T. Deduce that, for small enough values of p, testing no sticks minimises the expected net loss.

12 The random variable X has probability density function

$$f(x) = \begin{cases} kx^n(1-x) & 0 \leqslant x \leqslant 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where n is an integer greater than 1.

(i) Show that $k = (n+1)(n+2)$ and find μ, where $\mu = \mathrm{E}(X)$.

(ii) Show that μ is less than the median of X if

$$6 - \frac{8}{n+3} < \left(1 + \frac{2}{n+1}\right)^{n+1}.$$

By considering the first four terms of the expansion of the right-hand side of this inequality, or otherwise, show that the median of X is greater than μ.

(iii) You are given that, for positive x, $\left(1+\dfrac{1}{x}\right)^{x+1}$ is a decreasing function of x.

Show that the mode of X is greater than its median.

BLANK PAGE

BLANK PAGE

BLANK PAGE